

ON THE STABILITY OF THREE-LEVEL DIFFERENCE SCHEMES

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1970

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Abstract

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UDC 518.61:517.944/.947

MATHEMATICS

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ON THE STABILITY OF THREE-LEVEL DIFFERENCE SCHEMES

1. The general theory of stability of three-level schemes in the Hilbert space H_h was studied in ^(1,3). In ⁽¹⁾, a priori estimates were obtained expressing the stability of operator-difference three-level schemes in special (compound) norms containing the values of the solution on two neighboring layers. In ⁽³⁾, the necessity of the stability condition $\hat{R} > 1/4A$, obtained in ⁽¹⁾, was shown.

Here it will be shown that from the results of ⁽¹⁾ there follow a priori estimates for the solution in the usual energy norms $\bar{H}_A$ and $\bar{H}_B$ in terms of the right-hand sides in $H_{B^{-1}}$ and $H_{A^{-1}}$ (A and B are the operators of the three-level scheme, see (1)). A number of new a priori estimates is obtained.

2. We shall use the notation of ⁽²⁾: H_h is a real Hilbert space (with scalar product $(\cdot, \cdot)$ and norm $\|x\| = \sqrt{(x, x)}$), depending on the parameter h , an element of some normed space; $\omega_\tau = \{t_k = k\tau, k = 0, 1, \dots, k_0\}$ is a mesh with step τ on the interval $0 \leq t \leq t_0$; $y_{h\tau}(t_k)$, $\varphi_{h\tau}(t_k)$, etc. are abstract functions of the argument $t_k \in \omega_\tau$ with values in H_h ; $A_{h\tau}(t_k)$, $B_{h\tau}(t_k)$, etc. are linear operators specified on H_h . The indices h, τ are, as a rule, omitted.

The notation $A = A^* > 0$ means that A is a self-adjoint operator and $(Ax, x) > 0, \forall x \in H, x \neq 0$. If $(Ax, x) - (Bx, x) \geq 0, \forall x \in H$, then we write $A \geq B$, etc.

We shall also use the notation

$$y = y_k, \quad \hat{y} = y_{k+1}, \quad \check{y} = y_{k-1}, \quad y_t = (\hat{y} - y)/\tau, \quad y_{\bar{t}} = (y - \check{y})/\tau,$$

$$y_t^\circ = (\hat{y} - \check{y})/2\tau, \quad y_{t\bar{t}} = (\hat{y} - 2y + \check{y})/\tau^2, \quad \|y\|_A = \sqrt{(Ay, y)},$$

$$\|y\|_B = \sqrt{(By, y)}, \quad \|y\|_{B^{-1}} = \sqrt{(B^{-1}y, y)}.$$

3. Any three-level scheme can be written in the form

$$By_{t\bar{t}} + B_0 y_t^\circ + Ay = \varphi, \quad 0 < t = k\tau < t_0, \quad y(0) = y_0, \quad y_t(0) = \bar{y}_0, \quad (1)$$

where $y_0, \bar{y}_0$ are arbitrary elements of H ; $\varphi = \varphi(t)$ is an arbitrary function; $B = B(t)$, $A = A(t)$, $B_0 = B_0(t)$, $t \in \omega_\tau$.

In ⁽¹⁾, three-level schemes were studied that were written in one of the forms

$$By_{t\bar{t}} + \tau^2 R y_{t\bar{t}} + Ay = \varphi, \quad (2_1)$$

$$(E + \tau^2 R) y_{t\bar{t}} + B y_t^\circ + Ay = \varphi. \quad (2_2)$$

For convenience of exposition we are forced to change the notation and to write any three-level scheme in the canonical form (1). Comparing (1) with (2₁) and (2₂), we see that

$$B = \tau^2 R, \quad B_0 = B \text{ for } (2_1); \quad B = E + \tau^2 R, \quad B_0 = B \text{ for } (2_2).$$

Keeping these relations in mind, it is easy to use the results of ⁽¹⁾ for scheme (1).

The solution of problem (1) is the sum of the solutions of two problems

$$By_{t\bar{t}} + B_0 y_t^\circ + Ay = 0, \quad y(0) = y_0, \quad y_t(0) = \bar{y}_0; \quad (1a)$$

$$By_{t\bar{t}} + B_0 y_t^\circ + Ay = \varphi, \quad y(0) = 0, \quad y_t(0) = 0. \quad (1)$$

We shall assume that the original family of schemes (1) is specified by requirements on the operators

$$A = A^* > 0, \quad B = B^* > 0, \quad B_0 \geq 0, \quad \forall t \in \omega_\tau; \quad (3)$$

$$(1 - c_0 \tau) \check{A} \leq A \leq (1 + c_0 \tau) \check{A}, \quad (1 - c_0 \tau) \check{B} \leq \hat{B} \leq (1 + c_0 \tau) \check{B}, \quad (4)$$

where $c_0 = \text{const} > 0$ does not depend on h and τ . Conditions (4) mean that A and B satisfy Lipschitz conditions with respect to t . It follows from (1) that the condition

$$B \geq \frac{1 + \varepsilon}{4} \tau^2 A \quad \left(R \geq \frac{1 + \varepsilon}{4} A \text{ for } B = \tau^2 R \right), \quad (5)$$

where $\varepsilon = \text{const} > 0$ is an arbitrary constant independent of h and τ , is sufficient for the stability of scheme (1) with respect to the initial data, so that for scheme (1a) the estimate

$$\|Y_{k+1}\|_* \leq M_0 \|Y_1\|_*, \quad (6)$$

holds,

$$\|Y_{k+1}\|_*^2 = \frac{1}{4} \|y_{k+1} + y_k\|_{A(t_k)}^2 + ((B(t_k) - \frac{1}{4}\tau^2 A(t_k))y_{t,k}, y_{t,k}), \quad k > 0; \quad (7)$$

$$\|Y_1\|_*^2 = \frac{1}{4} \|y_1 + y_0\|_{A(\tau)}^2 + ((B(\tau) - \frac{1}{4}\tau^2 A(\tau))y_{t,0}, y_{t,0}), \quad (8)$$

$$\|y\|_{A(t_k)}^2 = (A(t_k)y, y),$$

$M_0 = M_0(c_0, \varepsilon) \geq 1$, $M_0 = 1$ if A and B are constant (not depending on t) operators, i.e. $c_0 = 0$.

4. Lemma 1. *If conditions (3) and (5) are satisfied, then*

$$\|Y_{k+1}\|_* \leq \|y_k\|_{A(t_k)} + \|y_{t,k}\|_{B(t_k)}; \quad (9)$$

$$\|Y_{k+1}\|_*^2 \geq \frac{\varepsilon}{1 + \varepsilon} \|y_{k+1}\|_{A(t_k)}^2; \quad (10)$$

$$\|Y_{k+1}\|_*^2 \geq \beta (\|y_{k+1}\|_{A(t_k)} + \|y_{t,k}\|_{B(t_k)})^2, \quad \beta = \frac{1}{4}\varepsilon/(1 + \varepsilon). \quad (11)$$

Proof. Denote

$$\hat{J} = \frac{1}{4} \|\hat{y} + y\|_A^2 + ((B - \frac{1}{4}\tau^2 A)y_t, y_t) = (Ay, \hat{y}) + \|y_t\|_B^2.$$

1) $\hat{J} = (Ay, y + \tau y_t) + \|y_t\|_B^2 = \|y\|_A^2 + \tau(Ay, y_t) + \|y_t\|_B^2 \leq \|y\|_A^2 + \tau\|y\|_A\|y_t\|_A + \|y_t\|_B^2$. By virtue of (5) we have

$$\tau\|y_t\|_A \leq \frac{2}{\sqrt{1 + \varepsilon}} \|y_t\|_B,$$

$$\hat{J} \leq \|y\|_A^2 + \frac{2}{\sqrt{1 + \varepsilon}} \|y\|_A\|y_t\|_B + \|y_t\|_B^2 \leq (\|y\|_A + \|y_t\|_B)^2.$$

Hence (9) follows.

$$\begin{aligned}
2) \quad \hat{J} &= (A\hat{y}, y) + \|y_t\|_B^2 = (A\hat{y}, \hat{y} - \tau y_t) + \|y_t\|_B^2 = \\
&= \|\hat{y}\|_A^2 - \tau(A\hat{y}, y_t) + \|y_t\|_B^2 \geq \|\hat{y}\|_A^2 - \tau\|\hat{y}\|_A\|y_t\|_A + \|y_t\|_B^2 \geq \\
&\geq \|\hat{y}\|_A^2 - \frac{2}{\sqrt{1+\varepsilon}}\|\hat{y}\|_A\|y_t\|_B + \|y_t\|_B^2, \\
\hat{J} &\geq (1 - c_0)\|\hat{y}\|_A^2 + \left(1 - \frac{1}{c_0(1+\varepsilon)}\right)\|y_t\|_B^2.
\end{aligned} \tag{12}$$

Choosing $c_0 = 1/(1 + \varepsilon)$, we obtain (10).

3) From (12) follows (11).

Substituting then (9) into (8), and (10) and (11) into (7), and using (6), we see that the following is valid.

Theorem 1. *Let conditions (3)–(5) be satisfied. Then scheme (1) is stable in H_A with respect to the initial data, so that for (1a) the estimates hold:*

$$\|y_{k+1}\|_{A(t_k)} \leq M_0 \sqrt{\frac{1+\varepsilon}{\varepsilon}} (\|y(0)\|_{A(\tau)} + \|y_t(0)\|_{B(\tau)}), \tag{13}$$

$$\|y_{k+1}\|_{A(t_k)} + \|y_t, k\|_{B(t_k)} \leq M_0 2 \sqrt{\frac{1+\varepsilon}{\varepsilon}} (\|y(0)\|_{A(\tau)} + \|y_t(0)\|_{B(\tau)}). \tag{13}$$

5. From the stability (13) with respect to the initial data of scheme (1) there follows its stability with respect to the right-hand side.

Theorem 2. *If conditions (3)–(5) are satisfied, then for the solution of problem (16) the a priori estimate holds*

$$\|y_{k+1}\|_{A(t_k)} \leq M_0 \sqrt{\frac{1+\varepsilon}{\varepsilon}} \sum_{j=1}^k \tau \|\varphi_j\|_{B^{-1}(t_j)}. \tag{14}$$

Proof. We seek the solution of problem (16) in the form

$$y_k = \sum_{j=1}^k \tau G_{k,j}, \quad k = 1, 2, \dots, \quad y_0 = 0, \tag{15}$$

where $G_{k,j}$, as a function of k , for any fixed $j = 1, 2, \dots$ satisfies equation (1a) and the initial conditions

$$G_{j,j} = 0, \quad (0.5\tau B_0(t_j) + B(t_j))(G_{j+1,j} - G_{j,j})/\tau = \varphi_j.$$

Hence, and from (15), it follows that $y(0) = y_t(0) = 0$. Since $B_0 \geq 0$, $B = B^* > 0$, then for the solution of the equation $(0.5\tau B_0 + B)w = \varphi$ we have $\|w\|_B \leq \|\varphi\|_{B^{-1}}$, i.e. $\|(G_t)_{j,j}\|_{B(t_j)} \leq \|\varphi_j\|_{B^{-1}(t_j)}$. By Theorem 1,

$$\|G_{k+1,j}\|_{A(t_k)} \leq M_0 \sqrt{(1+\varepsilon)/\varepsilon} \|(G_t)_{j,j}\|_{B^{-1}(t_j)}.$$

Using then the triangle inequality and (13),

$$\|y_{k+1}\|_{A(t_k)} \leq \sum_{j=1}^k \tau \|G_{k+1,j}\|_{A(t_k)},$$

we obtain (14).

6. Let us now consider the family of schemes

$$By_{\bar{t}t} + Ay = \varphi(t), \quad 0 < t = k\tau < t_0, \quad y(0) = y_0, \quad y_t(0) = \bar{y}_0. \quad (16)$$

We shall, as usual, denote by (16a) problem (16) with $\varphi = 0$, and by (16b) problem (16) with homogeneous initial data $y(0) = 0$, $y_t(0) = 0$. Suppose that

$$A \text{ and } B \text{ are constant operators, } A = A^* > 0, B = B^* > 0, \quad (17)$$

and condition (5) is valid.

Then scheme (16) is equivalent (cf. (2)) to the explicit scheme

$$x_{\bar{t}t} + Cx = \tilde{\varphi}, \quad 0 < t = \omega\tau, \quad x(0) = x_0, \quad x_t(0) = \bar{x}_0, \quad (18)$$

where $C = B^{-1/2}AB^{-1/2}$, $x = B^{1/2}y$, $\tilde{\varphi} = B^{-1/2}\varphi$. Let us rewrite it in the form

$$C^{-1}x_{\bar{t}t} + x = \Phi, \quad \Phi = C^{-1}\tilde{\varphi} = B^{1/2}A^{-1}\varphi, \quad (18^*)$$

and apply to it Theorems 1 and 2 (with $A = E$, $B_0 = 0$, $B = C^{-1}$),

$$\|x_{k+1}\| \leq \sqrt{\frac{1+\varepsilon}{\varepsilon}} \left[\|x_0\| + \|x_{t,0}\|_{C^{-1}} + \sum_{j=1}^k \tau \|\Phi_j\|_C \right],$$

taking into account that $M_0 = 1$. Since

$$\|x_k\| = \|y_k\|_B, \quad \|\Phi\|_C^2 = (C\Phi, \Phi) = (CC^{-1}\tilde{\varphi}, C^{-1}\tilde{\varphi}) = (C^{-1}\tilde{\varphi}, \tilde{\varphi}) = (B^{1/2}A^{-1}\varphi, B^{-1/2}\varphi) = (A^{-1}\varphi, \varphi),$$

$\|\Phi\|_C = \|\varphi\|_{A^{-1}}$, as a result, for (16) we obtain the estimate

$$\|y_{k+1}\|_B \leq \sqrt{\frac{1+\varepsilon}{\varepsilon}} \left[\|y(0)\|_B + \|By_t(0)\|_{A^{-1}} + \sum_{j=1}^k \tau \|\varphi_j\|_{A^{-1}} \right]. \quad (19)$$

Theorem 3. Let (5) and (17) be satisfied. Then for scheme (16) the a priori estimate (19) is valid.

Corollary. For an explicit scheme ($B = E$), under the condition $E \geq \frac{1}{4}(1 + \varepsilon)\tau^2 A$ ($\varepsilon > 0$ arbitrary), the estimate holds

$$\|y_{k+1}\| \leq \sqrt{\frac{1+\varepsilon}{\varepsilon}} \left[\|y(0)\| + \|y_t(0)\|_{A^{-1}} + \sum_{j=1}^k \tau \|\varphi_j\|_{A^{-1}} \right]. \quad (20)$$

Remark 1. It follows from (20) that the second initial value is taken in a weaker norm. Estimate (20) is convenient for investigating the rate of convergence in the grid norm L_2 of homogeneous difference schemes for equations with discontinuous coefficients.

Remark 2. If

$$B = E + \tau^2 R, \quad A \text{ and } R \text{ are constant operators, } R \geq 0, \quad (21)$$

then from (5), for problem (16), there follows the estimate

$$\|y_{k+1}\| \leq \sqrt{\frac{1+\varepsilon}{\varepsilon}} \sum_{j=1}^k \tau \|\varphi_j\|_{A^{-1}} \quad \text{for } B \geq \frac{1+\varepsilon}{4} \tau^2 A.$$

If, instead of (5), condition (see (1)) $R \geq \frac{1}{4}A$ is satisfied, then for (16) we obtain

$$\|y_{k+1}\| \leq \sqrt{2} t_{k+1} \sum_{j=1}^k \tau \|\varphi_j\|. \quad (22)$$

7. Estimate (20) can be obtained by the method of separation of variables in the case of finite-dimensional H and constant A .

For a variable operator $A = A(t)$, the method of separation of variables is ineffective.

If $A = A(t)$ is variable and B is a constant operator, then an estimate of the form (19) holds if

$$\rho^{-1} A \leq \hat{A} \leq \rho A, \quad \rho = e^{c_0 \tau} \quad \text{for all } t \in \omega_\tau, \quad (23)$$

where $c_0 = \text{const} > 0$ does not depend on τ and h . In this case, in (19) the right-hand side is multiplied by a constant $M = M(c_0) > 1$. Comparing (6)

with (23), we see that (23) follows from (6) for sufficiently small $\tau < \tau_0(c_0)$. In the case of scheme (1) of general form, Theorem 3 remains valid if $B_0 = B_0^*$ and the operators A, B, B_0 are permutable, while A and B are constant.

8. Let us consider, as an example, scheme (16) with weights (see (1)). For it,

$$B = E + \sigma\tau^2 A.$$

Condition (5) is satisfied for

$$\sigma \geq (1 + \varepsilon)/4 - 1/(\tau^2 \|A\|).$$

In particular, the explicit scheme ($\sigma = 0$) is stable if

$$\tau \leq \frac{2}{\sqrt{1 + \varepsilon}} \frac{1}{\sqrt{\|A\|}}.$$

Thus, for the equation of vibrations of a string,

$$\partial^2 u / \partial t^2 = \partial^2 u / \partial x^2, \quad 0 < x < 1, \quad t > 0, \quad u(0, t) = u(1, t) = 0,$$

$$u(x, 0) = u_0(x), \quad \partial u(x, 0) / \partial t = \bar{u}_0(x),$$

we have $A = -\Lambda$, $\Lambda y = y_{xx}$ (see (4)), and the explicit scheme is stable for

$$\tau \leq h / \sqrt{1 + \varepsilon},$$

where h is the grid step on the interval $0 \leq x \leq 1$.

Theorems 1, 2, 3 are applicable to the investigation and construction, by the regularization method (1), of difference schemes (in particular, economical schemes) for equations and systems of equations of mathematical physics.

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Received
24 XII 1969

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