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Abstract

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MATHEMATICS

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ON ONE CLASS OF INTEGRAL EQUATIONS OF CONVOLUTION TYPE AND ITS APPLICATION

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The following generalization of integral equations of convolution type is considered:

$$H\varphi \equiv a_0(t)\varphi(t) + \sum_{j=1}^n a_j(t) \int_{-\infty}^{\infty} b_j(\tau) h_j(t-\tau) \varphi(\tau) d\tau = f(t), \quad -\infty < t < \infty. \quad (1)$$

Here and everywhere below $h_j(t) \in \mathcal{L}_1(-\infty, \infty)$; $a_j(t)$ and $b_j(t)$ are bounded measurable functions, with $|a_0(t)| \geq c > 0$; $\varphi(t), f(t) \in E^*$.

Various special cases of equation (1) (mainly cases of piecewise-constant coefficients) have been studied, for example, in ⁽²⁻⁶⁾. The general equation of the form (1) with $b_j(t) \equiv 1$, $j = 1, 2, \dots, n$, was investigated by L. S. Rakovshchik ⁽¹⁾ under the assumption that $\varphi(t), f(t) \in \mathcal{L}_p(-\infty, \infty)$, $p > 1$.

In this note the results of ⁽¹⁾ are generalized to the operator H in the case of any of the spaces E . The results obtained are applied to the study of one class of integral equations of the form

$$K\psi \equiv \psi(x) + \sum_{j=1}^n c_j(x) \int_0^a d_j(y) k_j(x, y) \psi(y) dy = g(x), \quad 0 < x < a, \quad (2)$$

where $k_j(x, y)$ are homogeneous functions of arbitrary orders α_j ; $c_j(x)$, $d_j(x)$ belong to certain subclasses of bounded measurable functions. Necessary and sufficient conditions will be found for the Noether property of the operator K in a series of weighted spaces E_β , and the index will be computed. We note that equation (2) with $\alpha_j = -1$, $c_j(x), d_j(x) \in C(0, a)$ belongs to the class of equations studied by L. G. Mikhailov in ⁽¹²⁾.

No. 1. Let us introduce the spaces $M_0^{\sup}$, M_0^{mes} , $M^{\sup}$, M^{mes} of bounded measurable functions:

$$M^{\sup} = M_0^{\sup} \oplus X, \quad M^{\text{mes}} = M_0^{\text{mes}} \oplus X,$$

$$M_0^{\sup} = \{\varphi(t) \mid \varphi(t) \in M; \lim_{n \rightarrow \infty} \sup_{|t| > n} |\varphi(t)| = 0\},$$

$$M_0^{\text{mes}} = \{\varphi(t) \mid \varphi(t) \in M; \lim_{n \rightarrow \infty} \text{mes}(t : |\varphi(t)| > \varepsilon, |t| > n; \varepsilon > 0) = 0\},$$

where X is the two-dimensional space with basis $\theta(x)$, $\theta(-x)$, $\theta(x) = \frac{1}{2}(1 + \text{sign } x)$. Obviously, $\varphi(t) \in M^{\sup}$ (or M^{mes}) if and only if there exist constants $c_{\pm}$ such that $\theta(\pm t)[\varphi(t) - c_{\pm}] \in M_0^{\sup}$ (or M_0^{mes} , respectively). The constants $c_{\pm}$ will be called the (generalized) values of the bounded measurable function at infinity: $c_{\pm} = \varphi(\pm\infty)$ (cf. (1)).

*

We note that to the series E of Banach spaces $\mathcal{L}_p, M, M^u, M^c, C, C^0$ (7) we add the subspaces important for us, $M^{\sup}$, M^{mes} , of bounded measurable functions having (in a certain sense) values at infinity.

We note the following properties of the classes $M_0^{\sup}$, M_0^{mes} , $M^{\sup}$, M^{mes} :

1) they are complete in the norm M ; 2) functions from M^{mes} are continuous in measure on the whole axis; 3) $\varphi(t) \in M_0^{\sup}$ if and only if $\varphi(t)$ has a majorant from C^0 : $|\varphi(t)| \leq a(t) \in C^0$; 4) functions of bounded variation (on the closed axis) belong to $M^{\sup}$; 5) $M \cap \mathcal{L}_p \subset M_0^{\text{mes}}$ for any $p \geq 1$.

Everywhere below we shall assume that $E = \mathcal{L}_1, \mathcal{L}_p, M, M^{\sup}, M^{\text{mes}}, M_0^{\sup}, M_0^{\text{mes}}, M^u, M^c, C, C^0$, where the spaces $E = M^u, M^c, C, C^0$ of continuous functions will be understood in the following sense: $E = E_+ \oplus E_-$ (see (8)).

Lemma 1. If $h(t) \in \mathcal{L}_1$, then the convolution operator $A\varphi = h * \varphi$ acts from M_0^{mes} into C and from M_0^{mes} into C^0 .

No. 2. Put, without loss of generality, $a_0(t) \equiv 1$. Let $a_j(t), b_j(t) \in M^{\text{mes}}$. Represent the operator H in the form

$$H\varphi = \Pi\varphi + T_1\varphi + T_2\varphi, \quad (3)$$

where

$$\Pi\varphi = \begin{cases} \varphi(t) + \int_{-\infty}^{\infty} \sum_{j=1}^n a_j(\infty) b_j(\infty) h_j(t-\tau) \varphi(\tau) d\tau, & t > 0, \\ \varphi(t) + \int_{-\infty}^{\infty} \sum_{j=1}^n a_j(-\infty) b_j(-\infty) h_j(t-\tau) \varphi(\tau) d\tau, & t < 0. \end{cases}$$

is a paired operator, and

$$T_1\varphi = \sum_{j=1}^n \tilde{a}_j(h_j * b_j\varphi) + \sum_{j=1}^n (a_j - \tilde{a}_j)(h_j * \tilde{b}_j\varphi),$$

$$\begin{aligned} \tilde{a}_j(t) &= a_j(t) - a_j(\infty)\theta(t) - a_j(-\infty)\theta(-t); & \tilde{b}_j(t) &= b_j(t) - b_j(\infty)\theta(t) - \\ & & & - b_j(-\infty)\theta(-t), \end{aligned}$$

$$\begin{aligned} T_2\varphi &= \sum_{j=1}^n (b_j(\infty) - b_j(-\infty)) a_j(-\infty) \theta(-t) \int_{-\infty}^{\infty} \theta(\tau) h_j(t-\tau) \varphi(\tau) d\tau - \\ &- \sum_{j=1}^n (b_j(\infty) - b_j(-\infty)) a_j(\infty) \theta(t) \int_{-\infty}^{\infty} \theta(-\tau) h_j(t-\tau) \varphi(\tau) d\tau. \end{aligned}$$

Operators of the form T_2 are completely continuous ^(9,10) in all spaces E . For a broad class of coefficients this is also true for the operator T_1 . Namely, the following holds.

Lemma 2. *The operator $T\varphi = a(h * b\varphi)$, $h(t) \in \mathcal{L}_1$, $a(t), b(t) \in M$, is completely continuous in the space E , if:*

1) either

$$a(t) \in \begin{cases} M_0^{\text{mes}} & \text{for } E = \mathcal{L}_p \ (p \geq 1), \\ M_0^{\text{sup}} & \text{for } E = M, \ M_0^{\text{sup}}, \ M_0^{\text{mes}}, \ M^{\text{sup}}, \ M^{\text{mes}}, \\ C_0 & \text{for } E = M^u, \ M^c, \ C, \ C^0; \end{cases} \quad (4)$$

2) or

$$b(t) \in \begin{cases} M_0^{\text{sup}} & \text{for } E = \mathcal{L}_1, \\ M_0^{\text{mes}} & \text{for the remaining } E; \end{cases} \quad (5)$$

$$a(t) \in C, \quad \text{if } E = M^u, \ M^c, \ C, \ C^0.$$

For $b(t) \equiv 1$, $a(t) \in M_0^{\text{mes}}$, $E = \mathcal{L}_p$ ($p > 1$), the assertion of Lemma 2 was obtained in ⁽¹⁾.

By virtue of the complete continuity of the operators T_1, T_2 from (3), we obtain the following theorem:

Theorem 1. Let $h_j(t) \in \mathcal{L}_1$, $j = 1, 2, \dots, n$, and let $a_j(t)$, $j = 0, 1, \dots, n$, and $b_j(t)$, $j = 1, 2, \dots, n$, satisfy respectively the condi-

problems (4), (5), in which the index 0 is omitted*. Then the necessary and sufficient condition for the Noether property of the operator H in E has the form

$$\sigma(\lambda)_{\pm} = a_0(\pm\infty) + \sum_{j=1}^n a_j(\pm\infty)b_j(\pm\infty)\mathcal{H}_j(\lambda) \neq 0, \quad -\infty \leq \lambda \leq \infty,$$

where

$$\mathcal{H}_j(\lambda) = \int_{-\infty}^{\infty} h_j(t)e^{it\lambda} dt.$$

Under these conditions

$$\kappa_E(H) = -\frac{1}{2\pi} \Delta \left[\arg \frac{\sigma(\lambda)^+}{\sigma(\lambda)^-} \right]_{-\infty}^{\infty}.$$

Using one result from the theory of linear operators (⁽⁵⁾, Lemma 1), we arrive at the following theorem:

Theorem 2. The operator H has in all spaces E the same zeros and the same solvability conditions**, if one of the following conditions is fulfilled:

- 1) $a_j(t) \in M^{\text{mes}}$, $b_j(t) \in M^{\text{mes}}$, $E = \mathcal{L}_p$ ($p > 1$);
- 2) $a_j(t) \in M^{\text{mes}}$, $b_j(t) \in M^{\text{sup}}$, $E = \mathcal{L}_p$ ($p \geq 1$);
- 3) $a_j(t) \in M^{\text{sup}}$, $b_j(t) \in M^{\text{sup}}$, $E = \mathcal{L}_p$ ($p \geq 1$), M , M_0^{mes} , M_0^{sup} , M^{mes} , M^{sup} ;
- 4) $a_j(t) \in M^{\text{sup}}$, $b_j(t) \in M^{\text{mes}}$, $E = \mathcal{L}_p$ ($p > 1$), M , M_0^{mes} , M_0^{sup} , M^{mes} , M^{sup} ;
- 5) $a_j(t) \in C$, $b_j(t) \in M^{\text{mes}}$; E —arbitrary, except, perhaps, $\mathcal{L}_1$;
- 6) $a_j(t) \in C$, $b_j(t) \in M^{\text{sup}}$, E —arbitrary.

No. 3. In equation (2) we shall assume (cf. ⁽¹¹⁾) that there exists a number β such that

$$\int_0^\infty |k_j(1, y)| y^{-\beta} dy < \infty, \quad j = 1, 2, \dots, n. \quad (6)$$

Using a well-known device ([13, p. 396]), we reduce equation (2), by means of the transformation $x = ae^{-t}$, $y = ae^{-\tau}$, $\varphi(t) = e^{-\beta t}\psi(x)$, $f(t) = e^{-\beta t}g(x)$, to an equation on the semiaxis of the form (1). In this case

$$h_j(t) = a^{1+\alpha_j} e^{(1-\beta)t} k_j(1, e^t) \in \mathcal{L}_1,$$

$$a_j(t) = \theta(t) c_j(ae^{-t}) e^{-(1+\alpha_j)t}, \quad b_j(t) = \theta(t) d_j(ae^{-t}).$$

The indicated transformation establishes an isometry between the space $E(0, \infty)$ and the weighted space $E_\beta = E_\beta(0, a) = \{\psi \mid e^{-\beta t}\psi(ae^{-t}) = \varphi(t) \in E(0, \infty)\}$, $\|\psi\|_{E_\beta} = \|\varphi\|_{E_+}$. Obviously,

$$E_\beta = \begin{cases} x^{-\beta} E(0, a), & E \neq \mathcal{L}_p, \\ x^{-\beta+1/p} E(0, a), & E = \mathcal{L}_p, \end{cases}$$

and if U is the operator realizing the isometry of E_β onto E_+ , then $K = U^{-1} H U$.

We shall assume that in (2) $\psi(x), g(x) \in E_\beta$. Applying the results obtained for the operator H , we arrive at the following theorem:

Theorem 3. Let the kernels $k_j(x, y)$ be homogeneous functions of orders α_j , $-\infty < \alpha_j < \infty$, $j = 1, 2, \dots, n$, satisfying condition (6), and suppose also that $d_j(x) \in C(0, a)$, $x^{1+\alpha_j} c_j(x) \in C(0, a)$. Then the necessary and sufficient condition for the Noether property of the operator K in E_β has the form

$$\sigma(\lambda) = 1 + \sum_{j=1}^n a^{1+\alpha_j} d_j(0) \lim_{x \rightarrow 0} [x^{1+\alpha_j} c_j(x)] K_j(i\lambda - \beta + 1) \neq 0,$$

$$-\infty \leq \lambda \leq \infty,$$

* That is, the functions $\tilde{a}_j(t), \tilde{b}_j(t)$ satisfy the very same conditions (4), (5).

** The solvability conditions may be interpreted as orthogonality conditions to the zeros of the transposed operator, for example, in $\mathcal{L}_p$ ($p > 1$).

where

$$K_j(s) = \int_0^\infty k_j(1, x) x^{s-1} dx$$

is the Mellin transform of the function $k_j(1, x)$. If this condition is satisfied,

$$\kappa_{E_\beta}(K) = -\frac{1}{2\pi} \Delta[\arg \sigma(\lambda)]_{-\infty}^{\infty}.$$

Let us note that the formulation of Theorem 3, for simplicity, is given for the case of continuous coefficients $x^{1+\alpha_j} c_j(x)$, $d_j(x)$. It is not difficult to obtain a theorem for more general coefficients, for example, for $c_j(x)x^{1+\alpha_j}$, $d_j(x) \in M^{\sup}(0, a)$, where

$$M^{\sup}(0, a) = \{\psi(x) \mid \psi(x) \in M(0, a); \lim_{n \rightarrow \infty} \sup_{0 < x < 1/n} |\psi(x) - c| = 0\}, \quad c = \text{const.}$$

Similarly to Theorem 3, from Theorem 2 there follows a theorem on the zeros of the operator K , on whose formulation we shall not dwell. We also note that analogous theorems hold in the case $a = \infty$.

As an example, consider the equation

$$K_1 \psi \equiv \psi(x) + \int_0^a \sum_{j=1}^n k_j(x, y) \psi(y) dy = f(x), \quad 0 < x < a, \quad (7)$$

where $k_j(\lambda x, \lambda y) = \lambda^{\alpha_j} k_j(x, y)$. In the case $n = 1$ and $\alpha_1 = -1$ this equation was studied by L. G. Mikhailov ⁽¹¹⁾. Applying Theorem 3, we see that, for example, when $\alpha_j > -1$, $j = 1, 2, \dots, n$, equation (7) is Fredholm in all spaces E_β : $\kappa_{E_\beta}(K_1) = 0$ provided condition (6) is satisfied. It can be shown, using a theorem of L. Hörmander ⁽¹⁴⁾, p. 9, that the operator (7) is unbounded in any of the spaces E_β if $\alpha_j < -1$ at least for one j , $j = 1, 2, \dots, n$. And, finally, we note that when $n = 1$, $\alpha_j = \alpha \geq -1$, the zeros of the operator (7) have the form $\psi(x) = x^{1+\alpha-\beta} \psi_0(x)$, where $\psi_0(x)$ is continuous and $\psi(0) = 0$.

As a second example, consider the equation

$$K_2 \psi \equiv \psi(x) + c(x) \int_0^1 d(y) \frac{\ln y - \ln x}{y - x} \psi(y) dy = g(x), \quad 0 < x < 1,$$

in the spaces $\mathcal{L}_p$ ($1 < p < \infty$). Let $c(x), d(x) \in C(0, 1)$ and

$$v_p = \frac{1}{\pi} \sin \frac{\pi}{p}.$$

The necessary and sufficient condition for Noetherianity has the form

$$c(0)d(0) > -\frac{1}{\pi^2} = -v_2^2$$

for $p = 2$, and

$$c(0)d(0) \neq -v_p^2$$

for $p \neq 2$. In this case $\kappa_{\mathcal{L}_p}(K_2) = 0$ if $c(0)d(0) > -v_p^2$, $1 < p < \infty$, and

$$\kappa_{\mathcal{L}_p}(K_2) = \text{sign}(2 - p)$$

if $c(0)d(0) < -v_p^2$, $p \neq 2$.

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Note: Figure translations are in progress. See original paper for figures.

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