

ON THE CONTINUITY OF THE METRIC PROJECTION ON CERTAIN CLASSES OF SUBSPACES IN A BANACH SPACE

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Abstract

Full Text

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MATHEMATICS

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ON THE CONTINUITY OF THE METRIC PROJECTION ON CERTAIN CLASSES OF SUBSPACES IN A BANACH SPACE

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Let E and F be metric spaces, and let $K(F)$ be the set of all closed subsets of the space F . A mapping $\varphi : E \rightarrow K(F)$ is called a multivalued mapping from E into F . φ is called lower semicontinuous ⁽³⁾ if the set $\{x \in E : \varphi_x \subset G\}$ is open in E for every open subset $G \subset F$, and H -upper semicontinuous ^(4,5) if for every $x \in E$ the set φ_x is nonempty and the condition $x_n \rightarrow x$ implies the condition

$$\tilde{\rho}(\varphi_{x_n}, \varphi_x) = \sup\{\rho(y, \varphi_x) : y \in \varphi_{x_n}\} \rightarrow 0.$$

Both definitions coincide with the usual definition of continuity if φ is single-valued. If the mapping φ is upper semicontinuous and $\varphi_x \neq \emptyset$ for every $x \in E$, then φ will be H -upper semicontinuous; moreover, if φ_x is compact for every $x \in E$, then the converse is also true ⁽⁵⁾.

Let M be a set in E . The multivalued mapping T_M , which assigns to each point $x \in E$ the set

$$T_M x = \{y \in M : \rho(x, y) = \rho(x, M)\},$$

is called the metric projection of E onto M ⁽²⁾. The set M is called a set of existence ⁽¹⁾ if $T_M x \neq \emptyset$ for every $x \in E$, and Chebyshev ⁽¹⁾ if $T_M x$ is a singleton for every $x \in E$.

Everywhere in what follows: X is a B -space over the field of real numbers; $S = \{x \in X : \|x\| = 1\}$, $S^* = \{f \in X^* : \|f\| = 1\}$; $x_n \rightharpoonup x_0$ if the sequence $\{x_n\}$ converges weakly to x_0 ; $\overline{\text{sp}}\{f_n\}$ is the closed linear span of the sequence $\{f_n\} \subset X^*$; $\dim \text{sp}\{f_n\}$ is the dimension of the subspace $\text{sp}\{f_n\}$.

Theorem 1. *In a reflexive B -space X the following assertions are equivalent:*

- a) *the metric projection onto every closed subspace is lower semicontinuous;*
- b) *X satisfies the following condition (L): from $\{x_n\} \subset S$, $x_0 \in S$, $\{f_n\}_{n=0}^\infty \subset S^*$, $f_n(x_n) = f_0(x_0) = 1$, $x_n \rightharpoonup x_0$, $\|x_n - x_0\|_{\text{sp}\{f_n\}} \rightarrow 0$, where*

$$\|x_n - x_0\|_{\text{sp}\{f_n\}} = \sup\{f(x_n - x_0) : f \in \text{sp}\{f_n\} \cap S^*\},$$

it follows that $x_n \rightarrow x_0$;

- c) X satisfies the following condition (L.1): from $\{x_n\} \subset S$, $x_0 \in S$, $\{f_n\}_{n=0}^\infty \subset S^*$, $f_n(x_n) = f_0(x_0) = 1$, $x_n \rightharpoonup x_0$, $\rho(x_n - x_0, L) \rightarrow 0$, where $L = \bigcap_{n=0}^\infty \{x : f_n(x) = 0\}$, it follows that $x_n \rightarrow x_0$.

Corollary 1. In a reflexive B -space X satisfying condition (L) (or, what is the same, condition (L.1)), the metric projection onto every Chebyshev subspace is continuous.

Theorem 2. In order that in a reflexive B -space X the metric projection onto every closed subspace of finite defect-

then (respectively, onto every closed subspace of defect $\leq k$) is upper semi-continuous, it is necessary and sufficient that X satisfy the following condition (L^∞): from $\{x_n\} \subset S$, $x_0 \in S$, $\{f_n\}_{n=0}^\infty \subset S^*$, $f_n(x_n) = f_0(x_0) = 1$, $x_n \rightharpoonup x_0$, $\dim \text{sp}\{f_n\} < \infty$ it follows that $x_n \rightarrow x_0$ (respectively, that X satisfy the following condition L^k : from $\{x_n\} \subset S$, $x_0 \in S$, $\{f_n\}_{n=0}^\infty \subset S^*$, $f_n(x_n) = f_0(x_0) = 1$, $x_n \rightharpoonup x_0$, $\dim \text{sp}\{f_n\} \leq k$ it follows that $x_n \rightarrow x_0$).

Corollary 2. In a reflexive B -space X satisfying condition (L^∞) (respectively, satisfying condition (L^k)), the metric projection onto every Chebyshev subspace of finite defect (respectively, onto every Chebyshev subspace of defect $\leq k$) is continuous.

Theorem 3. In order that, in a B -space X , every closed subspace be Chebyshev with a continuous metric projection, it is necessary and sufficient that X be reflexive and satisfy the following condition (L_R): from $\{x_n\} \subset S$, $x_0 \in S$, $\{f_n\}_{n=0}^\infty \subset S^*$, $f_n(x_n) = f_0(x_0) = 1$, $\|x_n - x_0\|_{\text{sp}\{f_n\}} \rightarrow 0$ it follows that $x_n \rightarrow x_0$ (or, what is the same, condition

$$(L_R.1) : \quad \text{from } \{x_n\} \subset S, x_0 \in S, \{f_n\}_{n=0}^\infty \subset S^*, f_n(x_n) = f_0(x_0) = 1, \rho(x_n - x_0, L) \rightarrow 0,$$

where

$$L = \bigcap_{n=0}^\infty \{x : f_n(x) = 0\},$$

it follows that $x_n \rightarrow x_0$).

Theorem 4. In order that, in a strictly convex and reflexive B -space X , the metric projection onto every closed subspace of finite defect (respectively, onto every closed subspace of defect $\leq k$) be continuous, it is necessary and sufficient that X satisfy condition (L^∞) (respectively, that X satisfy condition (L^k)).

Theorem 5. In a reflexive B -space the metric projection onto every closed hyperplane is H -upper semicontinuous (in a strictly convex space, continuous).

Theorem 6. In a B -space X satisfying condition (L) , the metric projection onto every reflexive subspace is H -upper semicontinuous. In particular, the metric projection onto every reflexive Chebyshev subspace is continuous.

A B -space X is called compactly rotund $(^4, ^5)$ (respectively, weakly compactly rotund $(^4, ^5)$) if for every $f \in S^*$ the set $\{x : f(x) = 1\} \cap S$ is either empty or compact (respectively, it is either empty or weakly compact).

Theorem 7. In a weakly compactly rotund B -space X satisfying condition (L) , the metric projection onto every reflexive subspace is upper semicontinuous.

Theorem 8. In a B -space X , the metric projection onto every hyperplane that is an existence set is H -upper semicontinuous.

Theorem 9. In order that, in a B -space X , the metric projection onto every closed hyperplane be upper semicontinuous, it is necessary and sufficient that X be compactly rotund.

Example. Consider, in the space l_2 of real numerical sequences $x = \{\xi_i\}$ summable with square, the following norm, equivalent to the original one:

$$\|x\|_{S'} = \inf\{|\lambda| : x \in \lambda S'\},$$

where

$$S' = \left\{ x = \{\xi_i\} \in l_2 : \sum_{i=1}^{\infty} \frac{\xi_i^2}{i^2} \leq 1, \quad \sum_{i=1}^{\infty} \xi_i^2 \leq 3 \right\}.$$

Denote by X the space l_2 with the norm $\|x\|_{S'}$. It is not difficult to see that X is reflexive, strictly convex, and satisfies condition (L^∞) . Con-

Thus, by virtue of Theorem 4, in the space X the metric projection onto each closed subspace of finite defect is discontinuous. It can be shown, however, that in X the metric projection onto some convex closed set is discontinuous.

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Note: Figure translations are in progress. See original paper for figures.

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