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Abstract

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MATHEMATICS

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ON THE STABILITY OF DIFFERENCE SCHEMES WITH RESPECT TO RIGHT-HAND SIDES

In papers ^(1,2), necessary and sufficient conditions were obtained for stability with respect to the initial data of two-level difference schemes, defined as operator-difference equations on a family of Hilbert spaces $\{H_h\}$.

In the present paper estimates are constructed for certain energy norms of solutions of two-level difference schemes in terms of the initial data $y_0 \in H_h$ and the right-hand side $\varphi(t) \in H_h$, which coincide for $\varphi = 0$ with the estimates of ^(1,2).

To study stability with respect to the right-hand side, the method of separating stationary inhomogeneities is applied (see ^(3,4)), which makes it possible to remove certain restrictions (such, for example, as smallness of the time step τ) of paper ⁽⁵⁾, where stability was investigated by the method of energy inequalities. In addition, it proved possible to consider the case of non-self-adjoint operators and to establish a number of new estimates. We note that all the estimates obtained hold under the same conditions on the operators of the difference scheme that ensure stability with respect to the initial data.

1. Let $\{H_h\}$ be a family of Hilbert spaces (real or complex) depending on a parameter h (H_h is an analogue of the space of grid functions), $\omega_\tau = \{t_n = n\tau, n = 0, 1, \dots\}$ a grid with step $\tau > 0$, and $A, B : H_h \rightarrow H_h$ linear (additive and homogeneous) operators acting in H_h and depending on h , τ , and $t = t_n$.

A two-level difference scheme is defined as the operator equation (see ⁽⁵⁾)

$$B(t_n)y_t + A(t_n)y = \varphi(t_n), \quad n = 0, 1, \dots; \quad y_0 \in H_h, \quad (1)$$

where $y = y_n = y_{h,\tau}(t_n) \in H_h$, $y_t = (y_{n+1} - y_n)/\tau$, $\varphi(t) \in H_h$.

An operator $A : H \rightarrow H$ is called positive ($A > 0$) if $(Ax, x) > 0$ for all $0 \neq x \in H$, and nonnegative ($A \geq 0$) if $(Ax, x) \geq 0$ for all $x \in H$. If A is

a self-adjoint and positive operator, then one may introduce the energy norm $\|y\|_A = \sqrt{(Ay, y)}$.

The following lemmas will be needed for the proof of the stability theorems.

Lemma 1 (see ^(1,2)). The inequality

$$\|S\| \leq \rho, \quad S = E - \tau C, \quad (2)$$

follows from the condition

$$B_0 \geq \frac{\tau}{1+\rho} A, \quad B_0 = \operatorname{Re} B = \frac{B + B^*}{2}, \quad (3)$$

if $\rho \geq 1$, $A^* = A > 0$, $C = A^{1/2} B^{-1} A^{1/2}$, and from the condition

$$(A^{-1})_0 \geq \frac{\tau}{1+\rho} B^{-1}, \quad (A^{-1})_0 = \operatorname{Re} A^{-1}, \quad (4)$$

if $\rho \geq 1$, $B^* = B > 0$, $C = B^{-1/2} A B^{-1/2}$.

Remark. In the case of a real space H_h , inequalities (3) and (4) are equivalent respectively to the conditions

$$B \geq \frac{\tau}{1+\rho} A; \quad A^{-1} \geq \frac{\tau}{1+\rho} B^{-1}.$$

We shall say (see (5)) that the self-adjoint operator $\mathcal{D} = \mathcal{D}(t)$ satisfies the Lipschitz condition with respect to t if there exists a constant $c_1 \geq 0$, independent of τ, h, n , such that for all t and τ the operator inequalities

$$(1 - c_1 \tau) \mathcal{D}(t - \tau) \leq \mathcal{D}(t) \leq (1 + c_1 \tau) \mathcal{D}(t - \tau). \quad (5)$$

Lemma 2. If $\mathcal{D}(t)$ is a self-adjoint positive operator, then the Lipschitz condition (5) is equivalent to the estimate

$$\|\mathcal{D}^{1/2}(t)(\mathcal{D}^{-1}(t) - \mathcal{D}^{-1}(t - \tau))\mathcal{D}^{1/2}(t)\| \leq c_1 \tau. \quad (6)$$

Proof. Passing in (5) to inverse operators, we obtain

$$-c_1 \tau \mathcal{D}^{-1} \leq \mathcal{D}^{-1} - \widetilde{\mathcal{D}}^{-1} \leq c_1 \tau \mathcal{D}^{-1},$$

where $\mathcal{D} = \mathcal{D}(t)$, $\widetilde{\mathcal{D}} = \mathcal{D}(t - \tau)$. Hence the inequality

$$-c_1 \tau E \leq \mathcal{D}^{1/2}(\mathcal{D}^{-1} - \widetilde{\mathcal{D}}^{-1})\mathcal{D}^{1/2} \leq c_1 \tau E,$$

equivalent to (6), follows.

2. Theorem 1. Let the operator $A(t)$ satisfy the Lipschitz condition and let $A^*(t) = A(t) > 0$. Then, if the stability condition (3) is fulfilled with constant $\rho = e^{c_0\tau}$, $c_0 \geq 0$, for the solution of problem (1) the estimate

$$\|y_{n+1}\|_{A(t_n)} \leq \tilde{\rho}^{n+1} \|y_0\|_{A(0)} + \sum_{n'=0}^n \tau \tilde{\rho}^{n-n'} \|B^{-1} \varphi_{n'}\|_{A(t_{n'})}, \quad n = 0, 1, \dots, \quad (7)$$

is valid, where $\tilde{\rho} = e^{\tilde{c}_0\tau}$, $\tilde{c}_0 = c_0 + c_1/2$.

Proof. Represent equation (1) in the form of an equivalent explicit scheme (see (1))

$$x_{n+1} = S\bar{x}_n + \tau A^{1/2} B^{-1} \varphi_n, \quad (8)$$

where $x_{n+1} = A^{1/2}(t_n) y_{n+1}$, $\bar{x}_n = A^{1/2}(t_n) y_n$, $S = E - \tau C$, $C = A^{1/2} B^{-1} A^{1/2}$. From the stability condition (3), by Lemma 1, the estimate (2) of the norm of the transition operator S follows, and from the Lipschitz condition—the obvious estimate

$$\|\bar{x}_n\| \leq \sqrt{1 + c_1\tau} \|x_n\| \leq e^{c_1\tau/2} \|x_n\|.$$

Therefore from (8) we obtain

$$\|x_{n+1}\| \leq \tilde{\rho} \|x_n\| + \tau \|A^{1/2} B^{-1} \varphi_n\|,$$

whence (7) follows.

Similarly one proves

Theorem 2. Let the operator $B(t)$ satisfy the Lipschitz condition and let $B^*(t) = B(t) > 0$. Then, if the stability condition (4) is fulfilled with constant $\rho = e^{c_0\tau}$, $c_0 \geq 0$, for the solution of problem (1) the estimate

$$\|y_{n+1}\|_{B(t_n)} \leq \tilde{\rho}^{n+1} \|y_0\|_{B(0)} + \sum_{n'=0}^n \tau \rho^{n-n'} \|\varphi(t_{n'})\|_{B^{-1}(t_{n'})}, \quad (9)$$

where $\tilde{\rho} = e^{\tilde{c}_0\tau}$, $\tilde{c}_0 = c_0 + c_1/2$.

We note that if the operator $A(t)$ is also self-adjoint, then for estimate (9) condition (3) is sufficient.

3. Estimates different from (7) can be obtained by using the method of separating stationary inhomogeneities, which consists in the following.

Let the operator $A(t)$ in scheme (1) be self-adjoint and positive. Represent the solution of problem (1) as the sum

$$y_n = v_n + w_n, \quad n = 0, 1, \dots, \quad (10)$$

where w is the solution of the equation

$$A(t_n) w(t_{n+1}) = \varphi(t_n), \quad n = 0, 1, \dots \quad (11)$$

Suppose, moreover, that $w(0) = w(\tau)$. To determine the function $v(t)$, we obtain the problem

$$B(t)v_t + A(t)v = \tilde{\varphi}, \quad v_0 = y_0 - w_1, \quad (12)$$

where $\tilde{\varphi} = -(B - \tau A)w_t$.

The solution v_{n+1} of the auxiliary problem (12) is estimated by Theorem 1. After this, using the triangle inequality, from (10) and (11) we obtain an estimate for the solution of problem (1).

Theorem 3. *Under the same conditions as in Theorem 1, for the solution of problem (1) the estimate*

$$\begin{aligned} \|y_{n+1}\|_{A(t_n)} &\leq \tilde{\rho}^{n+1}(\|y_0\|_{A(0)} + \|\varphi(0)\|_{A^{-1}(0)}) + \|\varphi(t_n)\|_{A^{-1}(t_n)} \\ &\quad + \sum_{n'=1}^n \tau \tilde{\rho}^{n+1-n'} [c_1 \|\varphi_{n'-1}\|_{A^{-1}(t_{n'})} + \|\varphi_{n',\bar{t}}\|_{A^{-1}(t_{n'})}], \end{aligned} \quad (13)$$

holds, where $\tilde{\rho} = e^{\tilde{c}_0\tau}$, $\tilde{c}_0 = c_0 + c_1/2$, $\varphi_{n,\bar{t}} = (\varphi_n - \varphi_{n-1})/\tau$.

Proof. Using the method of separating stationary inhomogeneities, we obtain the estimate

$$\|y_{n+1}\|_{A(t_n)} \leq \|v_{n+1}\|_{A(t_n)} + \|\varphi(t_n)\|_{A^{-1}(t_n)}, \quad (1)$$

where v_{n+1} is the solution of problem (12). Just as in Theorem 1, for (12) the estimate

$$\|v_{n+1}\|_{A(t_n)} \leq \tilde{\rho} \|v_n\|_{A(t_{n-1})} + \tau \|B^{-1}\tilde{\varphi}_n\|_{A(t_n)}, \quad n = 1, 2, \dots, \quad (15)$$

can be proved, and

$$\|v_1\|_{A(0)} \leq \rho \|v_0\|_{A(0)}.$$

To estimate the term $\|B^{-1}\tilde{\varphi}_n\|_A$ in (15), note that

$$\|B^{-1}\tilde{\varphi}\|_A = \|SA^{1/2}w_t\|,$$

where $S = E - \tau C$, $C = A^{1/2}B^{-1}A^{1/2}$.

Further, according to (11) we have

$$w_t = (A^{-1}\varphi)_{\bar{t}} = A^{-1}\varphi_{\bar{t}} + (A^{-1} - \tilde{A}^{-1})\varphi(t_{n-1})/\tau,$$

so that the estimate

$$\|A^{1/2}w_t\| \leq \|\varphi_t\|_{A^{-1}(t_n)} + \frac{1}{\tau} \|A^{1/2}(A^{-1} - \check{A}^{-1})A^{1/2}\| \|\varphi(t_{n-1})\|_{A^{-1}(t_n)}$$

is valid.

Hence, taking Lemma 2 into account, we obtain

$$\|B^{-1}\tilde{\varphi}_n\|_{A(t_n)} \leq \rho\{c_1\|\varphi_{n-1}\|_{A^{-1}(t_n)} + \|\varphi_{n,\bar{t}}\|_{A^{-1}(t_n)}\}.$$

Thus, from (15) there follows the estimate

$$\|v_{n+1}\|_{A(t_n)} \leq \tilde{\rho}^{n+1}\|v_0\|_{A(0)} + \sum_{n'=1}^n \tau \rho^{n+1-n'} [c_1\|\varphi_{n'-1}\|_{A^{-1}(t_{n'})} + \|\varphi_{n',\bar{t}}\|_{A^{-1}(t_{n'})}].$$

Finally, noting that

$$\|v_0\|_{A(0)} \leq \|y_0\|_{A(0)} + \|w_1\|_{A(0)} = \|y_0\|_{A(0)} + \|\varphi_0\|_{A(0)},$$

we obtain (13).

4. In the case of permutable operators, more general estimates can be obtained.

Theorem 4. *Let the operator $A(t)$ satisfy the Lipschitz condition, $A^*(t) = A(t) > 0$, $AB = BA$, and $A(t)A(t-\tau) = A(t-\tau)A(t)$ for all t and τ . Then, if the stability condition (3) is fulfilled with $\rho = e^{c_0\tau}$, $c_0 \geq 0$, then for*

for $l = 0, 1, 2, \dots$ the following estimates hold for the solution of problem (1):

$$\|A^l(t_n)y_{n+1}\| \leq \tilde{\rho}^{n+1}\|A^l(0)y_0\| + \sum_{n'=0}^n \tau \tilde{\rho}^{n-n'} \|A^l B^{-1}\varphi_{n'}\|, \quad (16)$$

$$\begin{aligned} \|A^l(t_n)y_{n+1}\| &\leq \tilde{\rho}^{n+1}\|A^l(0)y_0\| + \|A^{l-1}(t_n)\varphi_n\| + \tilde{\rho}^{n+1}\|A^{l-1}(0)\varphi_0\| \\ &\quad + \sum_{n'=1}^n \tau \tilde{\rho}^{n+1-n'} [c_1\|A^{l-1}(t_{n'})\varphi_{n'-1}\| + \|A^{l-1}(t_{n'})\varphi_{n',\bar{t}}\|], \end{aligned} \quad (17)$$

where $\tilde{\rho} = e^{\tilde{c}_0\tau}$, $\tilde{c}_0 = c_0 + lc_1$.

Theorem 5. Suppose that the operator $B(t)$ satisfies the Lipschitz condition, $B^*(t) = B(t) > 0$, $AB = BA$, $B(t)B(t-\tau) = B(t-\tau)B(t)$. Then, if the stability

condition (4) is fulfilled with $\rho = e^{c_0\tau}$, $c_0 \geq 0$, then for $l = 0, 1, 2, \dots$, for the solution of problem (1) the estimate

$$\|B^l(t_n)y_{n+1}\| \leq \tilde{\rho}^{n+1} \|B^l(0)y_0\| + \sum_{n'=0}^n \tau \tilde{\rho}^{n-n'} \|B^{l-1}(t_{n'})\varphi_{n'}\| \quad (18)$$

is valid, where $\rho = e^{c_0\tau}$, $\tilde{c}_0 = c_0 + lc_1$.

Remark 1. If the operator $A(t)$ is also self-adjoint, then (18) holds under condition (3).

Remark 2. Estimates (16), (18) with $l = 0$ also hold without the Lipschitz-continuity requirement.

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Note: Figure translations are in progress. See original paper for figures.

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