

ON THE EXISTENCE OF WEIGHTED FINITE INCIDENCE STRUCTURES

A. Ya. Petrenyuk

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Abstract

Full Text

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MATHEMATICS

A. Ya. Petrenyuk

ON THE EXISTENCE OF WEIGHTED FINITE INCIDENCE STRUCTURES

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The article introduces the concept of a weight function of a finite incidence structure; with the aid of this concept, necessary conditions are found for the existence of l -weighted structures, defined below.

Let v be a natural number, $v \geq 4$, and let E be a set consisting of v elements. The symbol $P(E)$ denotes the collection of subsets of the set E . Let $n(\beta)$ be a function defined on $P(E)$ and taking values in the set Z^+ of nonnegative integers. This function defines on E a (finite) incidence structure (briefly: a structure), containing the subset $\beta \subseteq E$ exactly $n(\beta)$ times. The elements of the set $P(E)$ that enter the structure are called blocks, and the function $n(\beta)$ is called the function of block multiplicities in the structure. If H is a finite set, then $|H|$ often denotes the number of elements in H ; the number $|\beta|$ is called the volume of the block β .

Each structure defines on $P(E)$ a function

$$\rho(X) = \sum_{\{\beta: X \subseteq \beta\}} n(\beta), \quad (1)$$

which we shall call the weight function of the structure.

Let $K = \{k_1, \dots, k_s\}$ be a set of natural numbers; let l be a natural number, $2 \leq l < k_1 < \dots < k_s < v$; let $P_l(E) = \{L : L \subset E, |L| = l\}$, and suppose that on $P_l(E)$ a function $\lambda(L)$ with values in Z^+ is given. A structure for each block β of which the inclusion $|\beta| \in K$ holds and in which for every $L \in P_l(E)$ the equality

$$\rho(L) = \lambda(L), \quad (2)$$

holds is called an l -weighted structure of type $C(K, l, \lambda(L), E)$. The substructure A_i of an l -weighted structure A , defined by the block-multiplicity function

$$n_i(\beta) = \begin{cases} n(\beta), & \text{if } |\beta| = k_i, \\ 0 & \text{otherwise,} \end{cases}$$

is called the i -th equiblock component of the structure A ; the weight function of A_i is denoted by $\rho_i(X)$.

Counting in two different ways the number of occurrences of subsets D such that $T \subseteq D \subset E$, in the i -th equiblock component, we obtain that the equality

$$\binom{k_i - t}{d - t} \rho_i(T) = \sum_{\substack{H \subseteq E \setminus T \\ |H| = d - t}} \rho_i(T \cup H), \quad (3)$$

is valid, where d is a natural number, $0 \leq |T| = t < d \leq k_i$, $T \subset E$.

Putting $d = l$ in (3) and summing the equalities (3) with respect to the index i , taking into account (2) and the obvious formula

$$\rho(X) = \sum_{i=1}^s \rho_i(X),$$

we obtain the following theorem.

Theorem 1. *Suppose there is an l -weighted structure of type $C(K, l, \lambda(L), E)$. Then the relation*

$$\sum_{i=1}^s \binom{k_i - t}{l - t} \rho_i(T) = \sum_{\substack{H \subseteq E \setminus T \\ |H| = l - t}} \lambda(T \cup H) \quad (4)$$

holds for every T , $T \subset E$, $0 \leq |T| = t < l$.

We shall denote by

$$d \left[\binom{k_i - t}{l - t} \right]_{i=1}^s$$

the greatest common divisor of the integers

$$\binom{k_1 - t}{l - t}, \dots, \binom{k_s - t}{l - t}.$$

Theorem 2. *For the existence of an l -weighted structure of type $C(K, l, \lambda(L), E)$, it is necessary that the conditions*

$$\sum \lambda(T \cup H) / d \left[\binom{k_i - t}{l - t} \right]_{i=1}^s = \text{an integer} \quad (5)$$

hold for all T , $T \subset E$, $0 \leq |T| = t < l$. The summation in (5) is taken over all H , $H \subset E \setminus T$, $|H| = l - t$.

Condition (5) follows directly from (4).

In the case when the function $\lambda(L)$ is identically equal to the natural number λ , an l -weighted structure of type $C(K, l, \lambda, E)$ is called **l -balanced**. For $s = 1$ the latter is a tactical configuration ^(1,3), which for $l = 2$ is called a balanced incomplete block design ⁽²⁾. Condition (5) for the case of l -balanced structures takes the form

$$\lambda \binom{v-t}{l-t} / d \left[\binom{k_i-t}{l-t} \right]_{i=1}^s = \text{an integer}, \quad t = 0, 1, \dots, l-1. \quad (6)$$

Relation (4) for $\lambda(L) \equiv \lambda$, $l = 2$ and $t = 0$ occurs in ⁽²⁾. Conditions (6) for $s = 1$ turn into the known existence conditions for tactical configurations ⁽¹⁾.

Faculty of Mechanics and Mathematics
M. V. Lomonosov Moscow State University

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Note: Figure translations are in progress. See original paper for figures.

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