

ON SAMPLE MODELING OF THE GRAVITATIONAL FIELD OF THE SUN

A. Z. Petrov

1970

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-197001.54244>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 530.12

PHYSICS

A. Z. PETROV

ON SAMPLE MODELING OF THE GRAVITATIONAL FIELD OF THE SUN

(Presented by Academician V. A. Fock on 29 V 1969)

Let us consider the gravitational field in Einstein's theory. As was shown independently by A. Einstein ⁽¹⁾ and V. A. Fock ⁽²⁾, the equations of motion of test bodies in this theory follow from the field equations. Although the complete set of geodesics does not determine the metric (i.e., the field) exactly ^(3, Ch. 8), in some cases (the field in a vacuum, etc.) this correspondence is almost rigid: if the geodesics of two fields coincide under a mapping, then the fields differ little. It is possible and useful to generalize the problem. Let us subject the Riemannian space V_4 , which describes the gravitational field, to such a mapping onto some manifold $\bar{V}$ under which the geodesics of V_4 go over into curves of $\bar{V}$ determined by certain equations. We shall call this mapping the modeling of the field, V_4 the original, and $\bar{V}$ the model. In a number of papers ⁽⁴⁻⁶⁾ and others it has been shown that the problem admits a solution and is of fundamental interest.

1. The mathematical aspect of the problem leads to broad generalizations, but for physics the interesting method of modeling is that in which the model $\bar{V}$ is no more complicated than the original V_4 . Therefore, in what follows, as the model we choose $\bar{V}_4$ with metric of signature type $(---+)$, and we prescribe the equations of motion in the model in the form

$$\frac{\delta}{ds} \left(\frac{dx^\alpha}{ds} \right) = F^\alpha \left(x, \frac{dx}{ds} \right),$$

where $\frac{\delta}{ds}$ denotes covariant differentiation along the modeling curve. If one requires that all timelike and isotropic geodesics of V_4 be modeled, then we arrive at complete modeling; if only some set of geodesics of V_4 is modeled, we obtain sample modeling. In the first case it can be shown ⁽⁷⁾ that modeling of V_4 is possible in any $\bar{V}_4$ (even a flat one), with the "force" F^α being a polynomial of degree k with respect to dx^α/ds , and k is an even number ≤ 4 . The theory of sample modeling leads to nontrivial and as yet unsolved geometric problems. Without considering them, we shall confine ourselves below to the problem of sample modeling of the gravitational field of the Sun—the most interesting case,

connected with the effects of the perihelion shift of planets and the deflection of light rays.

Restricting ourselves to modeling the orbits of planets and the light rays touching the Sun, we obtain sample modeling; and by requiring that, in common coordinates, the modeled and modeling curves coincide with an accuracy determined by astronomical observations, we arrive at the problem of approximately sample modeling of the field of the Sun.

2. In Einstein' s theory of gravitation the field of the Sun is described by the Schwarzschild metric, which in polar coordinates has the form

$$(g_{\alpha\beta}) = \text{diag}(-\omega^{-1}, -r^2, -r^2 \sin^2 \theta, \omega),$$

where $\omega = 1 - 2\gamma M_{\odot}/c^2 r$, γ is the gravitational constant, $M_{\odot}$ is the mass of the Sun, and c is the speed of light. Since the choice of the model depends on us, we define it by the following requirements: 1) the model and the original are given in common coordinates; 2) the model V_4 is a flat space; 3) the equations of motion in

the models have the form $\delta/d\bar{s} = v_{\sigma}^{\alpha} dx^{\sigma}/d\bar{s}$, i.e., they reproduce, in outline, equations of the Lorentz type in electrodynamics; 4) the "field strengths" $v_{\sigma}^{\alpha}(x)$ possess the same symmetries as the Schwarzschild metric, since in the model it is precisely v_{σ}^{α} that determine the field; 5) just as in Einstein' s theory, we neglect the contribution to the field due to the rotation of the Sun.

In the terms of the Lie group theory of motions, the Schwarzschild metric admits a 4-member group of motions: $\mathfrak{L}_{\xi_i} g_{\alpha\beta} = 0$, where ξ_i^{α} are Killing vectors determining the 3-member group of rotations and the static character of the field, and $\mathfrak{L}_{\xi_i}$ is the Lie derivative along the direction ξ_i^{α} . In a polar coordinate system, ξ_i^{α} have the form

$$\xi_1^{\alpha} = \cos \varphi \delta_2^{\alpha} - \sin \varphi \text{ctg} \theta \delta_3^{\alpha}, \quad \xi_2^{\alpha} = -\partial_{\varphi} \xi_1^{\alpha}, \quad \xi_3^{\alpha} = -\delta_3^{\alpha}, \quad \xi_4^{\alpha} = \delta_4^{\alpha}.$$

Determining v_{σ}^{α} from the equations $\mathfrak{L}_{\xi_i} v_{\sigma}^{\alpha} = 0$ and using the 5th condition, we find that the only nonzero field strengths will be

$$v_4^1 = v_1^4 = B(r), \quad (1)$$

and the metric of the model will have the form

$$(\bar{g}_{\alpha\beta}) = \text{diag}(-1, -r^2, -r^2 \sin \theta, 1).$$

Wishing to obtain a model that physically interprets the one-center problem, one must assume that $B(r)$ has no singularities in the unclosed interval $(R_{\odot}, \infty)$, and

in this interval $B(r)$ admits an expansion in a Laurent series: $B(r) = \sum_{-\infty}^{+\infty} c_k r^k$. As $r \rightarrow \infty$, $B(r)$ must tend to zero, and consequently $c_k = 0$ for $k \geq 0$, and

$$B(r) = a_1/r + a_2/r^2 + a_3/r^3 + \dots \quad (2)$$

Let us determine a_1, a_2 from the correspondence principle. Consider a material point freely falling toward the Sun along the radius. In the polar coordinate system $\{r, \theta, \varphi, x^4 = ct\}$, along the trajectory of this point $\theta = \text{const}$, $\varphi = \text{const}$, and the first of the equations of motion will have the form

$$d^2 r / d\bar{s}^2 = (a_1/r + a_2/r^2 + a_3/r^3 + \dots) dx^4 / d\bar{s}.$$

The linear element is

$$d\bar{s}^2 = c^2 dt^2 \left\{ 1 - \frac{1}{c^2} \left(\frac{dr}{dt} \right)^2 \right\}.$$

Let $dr/dt \ll c$ for sufficiently large r , i.e. $d\bar{s} \simeq c dt$. We require that Newton's law of gravitation hold approximately. Then

$$\frac{1}{c^2} \frac{d^2 r}{d\bar{s}^2} = -\frac{\gamma M_{\odot}}{c^2 r^2} \simeq \frac{a_1}{r} + \frac{a_2}{r^2} + \dots,$$

i.e.,

$$a_1 = 0, \quad a_2 \stackrel{\text{def}}{=} -m = -\frac{\gamma M_{\odot}}{c^2} \simeq -1.47 \cdot 10^5 \text{ cm}$$

in the CGS system.

To determine a_3 , we use the more subtle effect of the displacement of the perihelion of planets. The curves modeling orbits are determined by the system of equations

$$\frac{d^2 r}{d\bar{s}^2} - r \left[\left(\frac{d\theta}{d\bar{s}} \right)^2 + \sin^2 \theta \left(\frac{d\varphi}{d\bar{s}} \right)^2 \right] = \left(-\frac{m}{r^2} + \frac{a_3}{r^3} + \dots \right) \frac{dx^4}{d\bar{s}};$$

$$\frac{d^2 \theta}{d\bar{s}^2} + \frac{2}{r} \frac{dr}{d\bar{s}} \frac{d\theta}{d\bar{s}} - \sin \theta \cos \theta \left(\frac{d\varphi}{d\bar{s}} \right)^2 = 0;$$

$$\frac{d^2 \varphi}{d\bar{s}^2} + \frac{2}{r} \frac{dr}{d\bar{s}} \frac{d\varphi}{d\bar{s}} + 2 \operatorname{ctg} \theta \frac{d\theta}{d\bar{s}} \frac{d\varphi}{d\bar{s}} = 0; \quad \frac{d^2 x^4}{d\bar{s}^2} = \left(-\frac{m}{r^2} + \frac{d_3}{r^3} + \dots \right) \frac{dr}{d\bar{s}},$$

$$-\left(\frac{dr}{ds}\right)^2 - r^2 \left[\left(\frac{d\theta}{ds}\right)^2 + \sin^2 \theta \left(\frac{d\varphi}{ds}\right)^2 \right] + \left(\frac{dx^4}{ds}\right)^2 = 1.$$

Using the possibility of choosing a polar coordinate system, we integrate the second equation with the initial data $\theta_0 = \pi/2$, $(d\theta/ds)_0 = 0$. If $\theta(\bar{s})$ is an analytic function, then, successively differentiating this equation and expanding $\theta(\bar{s})$ in a series, we find $\theta = \pi/2$. The third and fourth equations give the quadratures

$$\frac{d\varphi}{ds} = \frac{h}{r^2}, \quad \frac{dx^4}{ds} = q + \frac{m}{r} - \frac{d_3}{2r^2} + \dots,$$

as a consequence of which (and of the fifth equation) the first equation turns into an identity, while the fifth, after eliminating $d\bar{s}$ and making the inversion $r = 1/u$, takes the form

$$\left(\frac{du}{d\varphi}\right)^2 + u^2 = \frac{2m}{h^2}u + \frac{(m^2 - a_3)}{h^2}u^2 + O(u)^3,$$

i.e.

$$\frac{d^2u}{d\varphi^2} + u = \frac{m}{h^2} + \frac{(m^2 - a_3)}{h^2}u + O(u^2).$$

In Einstein's theory we have for planetary orbits the equation $d^2u/d\varphi^2 + u = m/h^2 + 3mu^2$. The constants h , q , characterizing the orbits, in both cases are determined from the classical approximation and are equal to $q = 1 + O(10^{-6})$ and $h \simeq 9.3 \cdot 10^8$ cm. As in Einstein's theory, we assume that the coordinates coincide with the astronomical ones, although there is something a posteriori in this assertion.

Let us apply to both equations simultaneously one and the same method of approximate integration. Discarding on the right all terms except the first, we obtain as the first approximation

$$u_1 = \frac{m}{h^2} \cdot [1 + \varepsilon \cos(\varphi - \varphi_0)],$$

where φ_0 is the longitude of perihelion, and ε is the eccentricity. Substituting this approximation into the right-hand sides of both equations and discarding insignificant or nonresonant corrections, we obtain on the right $\frac{m}{h^2} + \frac{6m^3}{h^4}\varepsilon \cos(\varphi - \varphi_0)$ in Einstein's theory and $\frac{m}{h^2} + \frac{m(m^2 - a_3)}{h^4}\varepsilon \cos(\varphi - \varphi_0)$ in our model. In order that the displacement of the perihelion of the planets in the original and in the model be the same, it is necessary and sufficient that $a_3 = -5m^2$ and

$v_4^1 = B(r) = -m/r^2 - 5m^2/r^3$. In approximate modeling the remaining terms of the expansion of $B(r)$ may be discarded.

3. A light ray can be modeled, with a corresponding choice of the parameter on the curve, in an isotropic or nonisotropic curve with the same equations of motion and the same tension as above. For example, repeating the integration of the system of equations of motion, under the condition of isotropy of the curve we arrive at the equation of a light ray in the model

$$\left(\frac{du}{d\varphi}\right)^2 + u^2 = \frac{q^2}{h^2} + \frac{2qm}{h^2}u + \frac{m^2(5q+1)}{h^2}u^2 + \frac{5m^3}{h^2}u^3 + \frac{25m^4}{4h^2}u^4.$$

If we take into account that $u \leq 1/R_\odot \sim 10^{-12}$, $mu \leq 10^{-7}$, and bear in mind the order of accuracy of the measurement of the deflection of a light ray, then the third, fourth, and fifth terms on the right may be discarded and the equation

$$\left(\frac{du}{d\varphi}\right)^2 + u^2 = \frac{q^2}{h^2} + \frac{2qm}{h^2}u,$$

where q, h are constants of integration depending on the shape and position of the light ray. We have the differential equation of a hyperbola (with eccentricity > 1), and, in order to determine from it the deflection of the light ray from a straight line, one must know two parameters q, h , in contrast to Einstein's theory, where only one parameter q/h appears. To determine q, h , we impose the conditions that hold in Einstein's theory (the original): 1) the orbits have perihelia $(du/d\varphi)|_{u=1/R_\odot} = 0$; 2) the deflection of the ray must give the Einstein value $\sim 4m/R_\odot$, or, what is the same thing, the mean observation of the deflection (~ 1.87). After this the model will be determined by the parameters $q \simeq m^2(\varepsilon^2 - 1)/R_\odot \simeq 0.47 \simeq 1/2$; $h = m\sqrt{\varepsilon^2 - 1} \simeq R_\odot/2$, and for such a hyperbola we obtain the deflection $\Delta\varphi - \pi = 4m/R_\odot$ with the accuracy allowed by observations. If the modeling curve is nonisotropic, then one must take $q \simeq 1.28$; $h \simeq 0.8R_\odot$.

4. Thus, Einstein's theory of the gravitational field of the Sun admits approximate selective modeling in a flat space, when the orbits of planets and light rays passing near the solar disk are modeled by non-geodesic curves determined by equations of motion of the Lorentz type in electrodynamics; and for this model the "field strengths," analogous to the tensor of the electromagnetic field, satisfy equations of the Maxwell type: $V_{[\alpha\beta,\gamma]} = 0$, $V^{\alpha\sigma}{}_{,\sigma} = j^\alpha$. The "current vector" j^α (without going into details) may be interpreted as the current vector of the gravitational field, by associating energy and the density of distributed mass with the field. We note that in such a model the "energy" of the gravitational field is written as a definite negative quantity (9)—a conclusion to which Maxwell also came, on the basis of other considerations; and the density of the "mass of the field" is likewise negative.

Such modeling is useful for some problems and deserves attention, without replacing Einstein's theory, but merely describing some of its results in other terms.

Kazan State University
named after V. I. Ulyanov-Lenin

Received
26 V 1969

CITED LITERATURE

- ¹ A. Einstein, L. Infeld, B. Hoffman. *Ann. Math.*, **39**, No. 1, 65 (1938).
- ² V. A. Fock, *ZhETF*, **9**, No. 4, 375 (1939).
- ³ A. Z. Petrov, *New Methods in the General Theory of Relativity*, "Nauka," 1966.
- ⁴ A. Z. Petrov, in: *Gravity and the Theory of Relativity*, Kazan, No. 4-5, 1968.
- ⁵ M. M. Kumar, *Curr. Sci. (India)*, **36**, No. 12, 313 (1967).
- ⁶ A. Z. Petrov, Abstracts GR-5, Tbilisi, 1968.
- ⁷ A. Z. Petrov, in: *Gravity and the Theory of Relativity*, No. 6, Kazan, 1969.
- ⁸ T. L. Synge, *Relativity. General Theory*, Amst., 1960.
- ⁹ A. Z. Petrov, K. A. Piragas, V. A. Dobrovolsky, in: *Gravity and the Theory of Relativity*, No. 4-5, Kazan, 1968.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.