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ON THE LOCAL STRUCTURE OF NORMAL FIELDS

E. I. Ostrovskii

1970

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Abstract

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UDC 519.21

MATHEMATICS

E. I. OSTROVSKII

ON THE LOCAL STRUCTURE OF NORMAL FIELDS

(Presented by Academician A. N. Kolmogorov, 15 IV 1970)

Let T be a separable compact topological space with the second axiom of countability; (Ω, B, P) the basic probability space; $x_t, t \in T$, a separable normal field whose correlation function $R(t, s)$ is continuous: $R : T \otimes T \rightarrow R^1$. It is also assumed that on T there exists a finite measure μ , regular with respect to the topology T ; we shall denote the topology by τ : $\mu(T) = 1$. C is the space of all continuous functions mapping T into R^1 , with norm $\forall f \in C$

$$\|f\| = \sup_{t \in T} |f(t)|;$$

L_2 is the space of functions quadratically summable with respect to μ , with norm

$$\|f\|_{L_2}^2 = \int_T f^2(t) d\mu(t).$$

Denote by λ_k and $\varphi_k(t)$, respectively, the eigenvalues and eigenfunctions of the operator $R : L_2 \rightarrow L_2$, given by the formula

$$(Rf)(t) = \int_T f(s) R(t, s) d\mu(s),$$

where $\|\varphi_n\|_{L_2} = 1$; by the theorems of Mercer and Hilbert-Schmidt,

$$\varphi_k \in C; \quad \lambda_k \geq 0; \quad (k = 1, 2, \dots),$$

$$R(t, s) = \sum_{k=1}^{\infty} \lambda_k \varphi_k(t) \varphi_k(s), \quad (1)$$

where the series in (1) converges absolutely and uniformly (see ⁽²⁾, p. 250). Put

$$\xi_k = \frac{1}{\sqrt{\lambda_k}} \int_T x_t \varphi_k(t) d\mu(t), \quad (2)$$

ξ_k are independent, normal with parameters $(0, 1)$. Denote

$$S_n(t) = \sum_{k=1}^n \sqrt{\lambda_k} \xi_k \varphi_k(t).$$

Theorem 1.

$$P \left\{ \forall t \in T, \lim_{n \rightarrow \infty} S_n(t) = x_t \right\} = 1. \quad (3)$$

Theorem 2. The set of discontinuity points of x_t is one and the same for almost all trajectories ⁽³⁾.

Theorem 3. In any neighborhood of a discontinuity point, with probability 1, the trajectory is unbounded.

The proof essentially repeats the analogous proof of Yu. K. Belyaev ⁽¹⁾, pp. 23–33), if instead of the spectral expansion one uses Theorem 1 and the following lemma.

Lemma 1. Let t_0 be a discontinuity point of x_t . Then there exists a nonrandom constant $a > 0$ such that

$$P \left\{ \overline{\lim}_{t \rightarrow t_0} |x_t - x_{t_0}| > 2a \right\} = 1.$$

Proof. By Theorem 2 the process x_t is discontinuous at t_0 with probability 1; this means that

$$P \left\{ \overline{\lim}_{t \rightarrow t_0} |x_t - x_{t_0}| = \varepsilon \right\} = 1, \quad \varepsilon > 0,$$

but then the quantity $\varepsilon = \varepsilon(\omega)$ is measurable with respect to the smallest σ -algebra generated by the quantities $\xi_k, \xi_{k+n}, \dots$, $\forall n \geq 1$, since $S_\xi(t)$ is continuous with probability 1; hence there exists a constant $a > 0$ such that $P\{\varepsilon = 2a\} = 1$, as was required to prove.

Corollary. Let G be a subgroup of the group of homeomorphisms of T onto itself such that for all $g \in G$

$$R(tg, t_1g) = R(t, t_1).$$

Then the set of discontinuity points of x_t is invariant with respect to G . If, moreover, T is full with respect to G , then the process x_t , if it is discontinuous, is unbounded in every open set.

To derive a necessary and sufficient condition for continuity, suppose that $T = [0, 2\pi]$; without essential loss of generality, $x_0 = x_{2\pi}$. Let $f(x) \in C$ and be periodic; put, for $p \geq 1$, p an integer,

$$\omega_p(f, \delta) = \sup_{|h| < \delta} \sup_x \left| \sum_{k=0}^p (-1)^{p-k} C_p^k f(x + kh) \right|,$$

$$K_n(x) = \frac{3}{2\pi n(2n^2 + 1)} \sin^4\left(\frac{nx}{2}\right) \bigg/ \sin^4\left(\frac{x}{2}\right)$$

(the Jackson kernel ⁽⁴⁾, pp. 140-145)

$$K_{mn}(x) = K_m(x) - K_n(x);$$

$$L_{mn}^k(t) = \int_{-\pi}^{\pi} e^{isk} K_{mn}(t-s) ds;$$

C^{2m+1} will denote the $(2m+1)$ -dimensional complex-number space with coordinates $\forall z \in C^{2m+1}$

$$z = (z_{-m}; z_{-m+1}; \dots z_{-1}; z_0; z_1; \dots z_m),$$

and define

$$(z, z) = \sum_{q=-m}^m |z_q|^2; \quad dz = \prod_{q=-m}^m d \operatorname{Re} z_q d \operatorname{Im} z_q.$$

The Bernstein function $\Phi(z)$ is defined for $z \in C^{2m+1}$ by the equality (see ⁽⁵⁾, p. 127 ff.)

$$\Phi(z) = \sup_x \left| \sum_{k=-m}^m z_k e^{ikx} \right|.$$

$\tilde{R}_m^n$ is a square matrix of order $(2m+1; 2m+1)$ with elements

$$(\tilde{R}_m^n)_{kl} = \int_0^{2\pi} \int_0^{2\pi} L_{mn}^k(t) \overline{L_{mn}^l(s)} R(t, s) dt ds$$

and R_m^n is the Hermitian square root of $\tilde{R}_m^n$. Also put, for $m > n$,

$$\tau_m^n = (2\pi)^{-2m} \int_{C^{2m+1}} \operatorname{arctg} \Phi(R_m^n z) e^{-1/2(z, z)} dz.$$

From known theorems of constructive function theory one can derive the following result ([4], pp. 144-145).

Theorem 4. *In order that x_t be continuous with probability 1, it is necessary and sufficient that*

$$\lim_{n \rightarrow \infty} \sup_{m > n} \tau_m^n = 0.$$

Theorem 5. Suppose that the condition of Theorem 4 is satisfied; then there exist constants C_p, d_p such that $\forall p \geq 1$

$$C_p \sup_{m > n} \tau_m^n \leq M \operatorname{arctg} \omega_p \left(x_t, \frac{1}{n} \right) \leq \frac{d_p}{n^p} \sum_{\nu=1}^n \nu^{p-1} \sup_{m > \nu} \tau_m^\nu.$$

By the indicated method one easily obtains the known sufficient conditions for continuity of the process x_t , if one uses the known Bernstein inequality ([4], p. 146).

Put

$$\tilde{\omega}_R(\delta) = \sup_{\substack{|h_1| \leq \delta \\ |h_2| \leq \delta}} \sqrt{|R(t+h_1, s+h_2) - R(t, s)|}.$$

Theorem 6 (Fernique, [6]). If

$$\int_0^\infty \tilde{\omega}_R(e^{-x^2}) dx < \infty,$$

then x_t is continuous with probability 1.

Corollary. Let $\omega_1(\delta)$ be a modulus of continuity such that

$$\lim_{\delta \downarrow 0} \frac{\omega_1(\delta)}{\delta} = \infty.$$

Then there exists a stationary process x_t such that for its correlation function $r(t) = Mx_s x_{s+t}$, $\omega_1[r(t), \delta] = \omega_1(\delta)$, but at the same time $P\{x_t \in G\} = 1$; and even $M \exp[\varepsilon \|x_t\|^2] < \infty$ for some $\varepsilon > 0$.

Theorems 4 and 5 have also been obtained for finite-dimensional fields.

The author takes this opportunity to express gratitude to S. A. Molchanov for supervising the work and to Yu. K. Belyaev for assistance and valuable suggestions.

Moscow State University
named after M. V. Lomonosov

Received
7 III 1970

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Note: Figure translations are in progress. See original paper for figures.

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