

ON THE QUESTION OF GENERALIZING THE INTEGRAL FORMULAS OF CAUCHY, SCHWARZ, AND POISSON

I. I. Bavrin

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Abstract

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MATHEMATICS

I. I. BAVRIN

ON THE QUESTION OF GENERALIZING THE INTEGRAL FORMULAS OF CAUCHY, SCHWARZ, AND POISSON

(Presented by Academician M. A. Lavrent'ev on March 6, 1970)

M. M. Dzhrbashyan ⁽¹⁾ established generalized formulas of Cauchy, Schwarz, and Poisson associated with a given function $\omega(x) \in \Omega$. Subsequently the author ⁽²⁾ obtained generalized formulas of Cauchy, Schwarz, and Poisson associated with a given system of functions $\omega_j(x) \in \Omega$ ($j = 1, \dots, m$). In the present note a substantial generalization of these latter formulas is given (Theorems 1 and 2).

1. Let the functions $\omega_j(x) \in \Omega$ ($j = 1, 2, \dots, m$), $\tilde{\omega}_{\tilde{j}}(x) \in \Omega$ ($\tilde{j} = 1, 2, \dots, \tilde{m}$). Suppose, further, that

$$p_j(0) = 1, \quad \tilde{p}_{\tilde{j}}(0) = 1,$$

$$p_j(r) = r \int_r^1 \frac{\omega_j(x)}{x^2} dx, \quad \tilde{p}_{\tilde{j}}(r) = r \int_r^1 \frac{\tilde{\omega}_{\tilde{j}}(x)}{x^2} dx \quad (r \in (0, 1)),$$

$$\Delta_0^{(j)} = 1, \quad \tilde{\Delta}_0^{(\tilde{j})} = 1, \quad \Delta_k^{(j)} = -(k+1) \int_0^1 r^k dp_j(r) = k \int_0^1 r^{k-1} \omega_j(r) dr,$$

$$\tilde{\Delta}_k^{(\tilde{j})} = -(k+1) \int_0^1 r^k d\tilde{p}_{\tilde{j}}(r) = k \int_0^1 r^{k-1} \tilde{\omega}_{\tilde{j}}(r) dr$$

$$(j = 1, 2, \dots, m; \quad \tilde{j} = 1, 2, \dots, \tilde{m}), \quad k = 1, 2, \dots, *$$

2. In ⁽²⁾ the author introduced into consideration the function

$$C(z; \omega_1, \dots, \omega_m) = \sum_{k=0}^{\infty} \frac{z^k}{\Delta_k^{(1)} \dots \Delta_k^{(m)}},$$

which is holomorphic in the disk $|z| < 1$. Therefore, by virtue of Theorem 1 of (2), the function

$$L_{(\tilde{\omega}_1, \dots, \tilde{\omega}_{\tilde{m}})} [C(re^{i\varphi}; \omega_1, \dots, \omega_m)] \equiv C_{(\tilde{\omega}_1, \dots, \tilde{\omega}_{\tilde{m}})}(re^{i\varphi}; \omega_1, \dots, \omega_m) =$$

$$= \sum_{k=0}^{\infty} \frac{\tilde{\Delta}_k^{(1)} \dots \tilde{\Delta}_k^{(\tilde{m})}}{\Delta_k^{(1)} \dots \Delta_k^{(m)}} (re^{i\varphi})^k, \quad (1)$$

* In (1) the function

$$p(r) = r \int_r^1 \frac{\omega(x)}{x^2} dx \quad (\omega(x) \in \Omega), \quad r \in (0, 1), \quad p(0) = 1,$$

and the sequence of numbers

$$\Delta_k = -(k+1) \int_0^1 r^k dp(r) \quad (k = 0, 1, 2, \dots)$$

were introduced; moreover, it was shown that all the numbers Δ_k ($k = 0, 1, 2, \dots$) are positive, with

$$\Delta_0 = 1, \quad \Delta_k = k \int_0^1 \omega(x) x^{k-1} dx \quad (k = 1, 2, \dots).$$

which, for brevity, we shall denote by $C_{(\tilde{\omega})}(re^{i\varphi}; \omega)$ ($\omega = (\omega_1, \dots, \omega_m)$, $\tilde{\omega} = (\tilde{\omega}_1, \dots, \tilde{\omega}_{\tilde{m}})$, and similarly everywhere below), is holomorphic in the disk $|z| < 1$.

Introduce the function

$$S_{(\tilde{\omega})}(z; \omega) = 2C_{(\tilde{\omega})}(z; \omega) - C_{(\tilde{\omega})}(0; \omega) = 1 + 2 \sum_{k=1}^{\infty} \frac{\tilde{\Delta}_k^{(1)} \dots \tilde{\Delta}_k^{(\tilde{m})}}{\Delta_k^{(1)} \dots \Delta_k^{(m)}} z^k, \quad (2)$$

noting that

$$C_{(\tilde{\omega})}(0; \omega) := \tilde{\Delta}_0^{(1)} \dots \tilde{\Delta}_0^{(\tilde{m})} / \Delta_0^{(1)} \dots \Delta_0^{(m)} = 1.$$

Let the function

$$f(re^{i\varphi}) = \sum_{k=0}^{\infty} a_k (re^{i\varphi})^k$$

be holomorphic in the disk $|z| < R$. Then, by ((2), Theorem 1), the function

$$L^{(\omega_1, \dots, \omega_m)}[f(re^{i\varphi})] \equiv f_{(\omega_1, \dots, \omega_m)}(re^{i\varphi}) = \sum_{k=0}^{\infty} \Delta_k^{(1)} \dots \Delta_k^{(m)} a_k (re^{i\varphi})^k$$

is holomorphic in the same disk $|z| < R$. Hence it follows that the function

$$f_{(\omega)}(z; \tilde{\omega}) := f_{(\omega_1, \dots, \omega_m)}(z; \tilde{\omega}_1, \dots, \tilde{\omega}_m) = \sum_{k=0}^{\infty} \frac{\Delta_k^{(1)} \dots \Delta_k^{(m)}}{\tilde{\Delta}_k^{(1)} \dots \tilde{\Delta}_k^{(m)}} a_k z^k \quad (3)$$

is also holomorphic in the disk $|z| < R^*$. Finally, relying on the expansions (1)–(3), we arrive at the following theorem.

Theorem 1. If the function

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

is holomorphic in the disk $|z| < R$, then for any ρ ($0 < \rho < R$) the integral formulas

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} C_{(\tilde{\omega})} \left(e^{-i\theta} \frac{z}{\rho}; \omega \right) f_{(\omega)}(\rho e^{i\theta}; \tilde{\omega}) d\theta \quad (|z| < \rho),$$

$$f(z) = i \operatorname{Im} f(0) + \frac{1}{2\pi} \int_0^{2\pi} S_{(\tilde{\omega})} \left(e^{-i\theta} \frac{z}{\rho}; \omega \right) \operatorname{Re} f_{(\omega)}(\rho e^{i\theta}; \tilde{\omega}) d\theta \quad (|z| < \rho).$$

3. Introduce the function

$$P_{(\tilde{\omega})}(\theta, r; \omega) = \operatorname{Re} S_{(\tilde{\omega})}(\rho e^{i\theta}; \omega) = 1 + 2 \sum_{k=1}^{\infty} \frac{\tilde{\Delta}_k^{(1)} \dots \tilde{\Delta}_k^{(m)}}{\Delta_k^{(1)} \dots \Delta_k^{(m)}} r^k \cos k\theta,$$

which is harmonic in the unit disk $0 \leq r < 1$, $0 \leq \theta \leq 2\pi$.

Let the function

$$u(re^{i\varphi}) = \alpha_0 + \sum_{k=1}^{\infty} (\alpha_k \cos k\varphi - \beta_k \sin k\varphi) r^k$$

($\alpha_0, \alpha_k, \beta_k$ ($k = 1, 2, \dots$) are real numbers**) be harmonic in the disk $|z| < R$. Then, by virtue of the author's theorem (Theorem 2 from (?)), the function

$$L^{(\omega_1, \dots, \omega_m)}[u(re^{i\varphi})] \equiv u_{(\omega_1, \dots, \omega_m)}(re^{i\varphi}) = \alpha_0 + \sum_{k=1}^{\infty} (\Delta_k^{(1)} \dots \Delta_k^{(m)} \alpha_k \cos k\varphi - \\ - \Delta_k^{(1)} \dots \Delta_k^{(m)} \beta_k \sin k\varphi) r^k$$

* The Cauchy-Hadamard formula is used.

** They remain so in what follows as well.

will be harmonic in the same disk $|z| < R$. Hence it follows that the function

$$u_{(\omega)}(re^{i\varphi}; \tilde{\omega}) = u_{(\omega_1, \dots, \omega_m)}(re^{i\varphi}; \tilde{\omega}_1, \dots, \tilde{\omega}_m) = \\ = \alpha_0 + \sum_{k=1}^{\infty} \left(\frac{\Delta_k^{(1)} \dots \Delta_k^{(m)}}{\tilde{\Delta}_k^{(1)} \dots \tilde{\Delta}_k^{(m)}} \alpha_k \cos k\varphi - \frac{\Delta_k^{(1)} \dots \Delta_k^{(m)}}{\tilde{\Delta}_k^{(1)} \dots \tilde{\Delta}_k^{(m)}} \beta_k \sin k\varphi \right) r^k$$

is also harmonic in the disk $|z| < R$.* Now Theorem 1 easily implies

Theorem 2. If the function

$$u(re^{i\varphi}) = \alpha_0 + \sum_{k=1}^{\infty} (\alpha_k \cos k\varphi - \beta_k \sin k\varphi) r^k$$

is harmonic in the disk $|z| < R$, then for every ρ ($0 < \rho < R$) the integral formula

$$u(re^{i\varphi}) = \frac{1}{2\pi} \int_0^{2\pi} P_{(\tilde{\omega})} \left(\varphi - \theta, \frac{r}{\rho}; \omega \right) u_{(\omega)}(\rho e^{i\theta}; \tilde{\omega}) d\theta \\ (0 \leq r < \rho, \quad 0 \leq \varphi \leq 2\pi).$$

is valid.

Moscow Regional Pedagogical Institute
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CITED LITERATURE

¹ M. M. Dzhrbashyan, *Izv. AN SSSR, ser. matem.*, **32**, No. 5, 1075 (1968). ²
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* The Cauchy-Hadamard formula is used.

Note: Figure translations are in progress. See original paper for figures.

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