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Abstract

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MATHEMATICS

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A PROBABILISTIC CHARACTERIZATION OF ZERO-DIMENSIONAL COMPACT GROUPS

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The basic convergence principle for compositions of non-identical distributions $\{\mu_m\}$ on a compact group G with a countable base asserts that the limiting points of the products $\mu_1 \dots \mu_n$, $n = 1, 2, \dots, \infty$, differ by right shifts by elements of G (¹). In other words, there exist elements α_n of G such that the measures $\mu_1 \dots \mu_n \alpha_n$ converge as $n \rightarrow \infty$. This assertion can be strengthened somewhat by choosing α_n , with a certain margin, so that for any i the sequence $\mu_i \mu_{i+1} \dots \mu_n \alpha_n$ converges as $n \rightarrow \infty$. The element α_n depends not only on the measure μ_n , but also on the entire sequence $\{\mu_m\}$. The question is whether α_n can be chosen to depend only on the measure μ_n .

Definition. A set of measures A on a compact group G will be called **complete** if, for any distributions $x_i \in A$, $i = 1, 2, \dots, \infty$, (the measures x_i may coincide) the composition $x_1 x_2 \dots x_n$ converges as $n \rightarrow \infty$.

It follows from the definition that for any $x \in A$, the power x^n converges. Hence there follows the existence of classes A , for example consisting of the powers of a measure x for which x^n converges.

Now let μ be an arbitrary measure on G . Form the shifts of the measure $\{\mu c\}$, where c runs through all elements of G . We shall call this collection of shifts the class of shifts of the measure μ . Obviously, classes of shifts either do not intersect, or coincide.

Take one representative from each class of shifts and form from them the set A . Can the representatives be chosen so that the set A is complete? If this is possible on a certain group G , then the problem of convergence of compositions of non-identical measures on G is completely solved. Indeed, if from the class of shifts containing the measure μ one representative is chosen, denote it by $\mu c(\mu)$, where $c(\mu) \in G$ and is determined by the class of shifts containing the measure μ . Then for the sequence $\{\mu_n\}$ put $\alpha_1 = c(\mu_1)$, $\alpha_2 = c(\alpha_1^{-1} \mu_2)$, $\dots$, $\alpha_{n+1} = c(\alpha_n^{-1} \mu_{n+1})$. By virtue of the completeness of A , the product $\mu'_1 \dots \mu'_n = \mu_1 \dots \mu_n \alpha_n$, where $\mu'_i = \alpha_{i-1}^{-1} \mu_i \alpha_i$, converges as $n \rightarrow \infty$. That is, the completeness of A is a stronger property of measures on the group G than

the convergence principle, and guarantees the existence of α_n depending only on the single measure μ_n .

Furthermore, the existence of the convergence principle for all measures on G is equivalent to the compactness of G ⁽¹⁾. In this note we give a probabilistic characterization of zero-dimensional compact groups. Theorem 3 shows that if G is a positive-dimensional compact group, then it is impossible to make the set A complete under any choice of representatives from the classes of shifts. Conversely, if G is zero-dimensional, then this can already be done (Theorem 2).

Let B be an arbitrary complete set of measures. The following several assertions concerning B follow from the definition of completeness.

Lemma 1. If B is complete, then the semigroup of measures generated by B is also complete.

Lemma 2. If B is complete, then the closure of B in the weak topology of measures is also complete.

Lemma 3. If B is complete, then $d^{-1}Bd$ is complete for any d in G .

From Lemmas 1 and 2 it follows:

Theorem 1. If a set of measures B on a compact group G is complete, then the smallest closed semigroup containing the set B is complete as well.

Theorem 2. If a compact group G is zero-dimensional, then one can compose a complete set of distributions A from representatives of each class of translates of measures on G .

Proof. First let the group G be finite, consisting of s elements, and let μ be some measure on it. Then there is an element e whose probability in the measure μ is not less than $1/s$. From the class of translates containing the measure μ , take the distribution

$$\hat{\mu} = \mu e^{-1}.$$

The collection $\{\hat{\mu}\}$ over all possible classes forms a complete set. Indeed, for any sequence $\{\hat{\mu}_i\}$ from this set, the probability of the identity G in each measure is not less than $1/s$. Therefore, by virtue of ⁽²⁾, the product

$$\hat{\mu}_1 \dots \hat{\mu}_n$$

converges as $n \rightarrow \infty$, i.e., the theorem is proved for finite groups.

Now let G be an arbitrary zero-dimensional compact group with an infinite number of elements. Then, according to ⁽³⁾, the group G is represented by a convergent sequence of finite groups

$$G_1, G_2, \dots, G_i, \dots; \quad i = 1, 2, \dots, \infty,$$

i.e., homomorphisms φ_i of the group G_{i+1} onto G_i are given, and the group G then proves to be isomorphic to G_ω , the group of sequences

$$(x_1, x_2, \dots, x_n)$$

with componentwise multiplication, where $x_i \in G_i$ and

$$x_i = \varphi_i \varphi_{i+1} \dots \varphi_{j-1}(x_j).$$

A base of neighborhoods of G_ω consists of sets of the form $[U_\alpha]$, where U_α is a neighborhood in G_α , and $[U_\alpha]$ is the collection of sequences in which the coordinates with number α belong to U_α . Denote the number of elements of G_i by

$$s_i = |G_i|.$$

Then $s_i \leq s_{i+1}$. Since G is infinite, G_ω is also infinite, and therefore $s_i \rightarrow \infty$.

By virtue of the isomorphism of G_ω and G , it is enough to prove Theorem 2 for the group G_ω . Let μ be a measure on G_ω . If $b_\alpha \in G_\alpha$, then the measure

$$\mu_\alpha(b_\alpha) = \mu[b_\alpha]$$

defines a measure on G_α and is called the projection of μ onto G_α .

Define an element

$$b = (b_1, b_2, \dots)$$

of G_ω so that the probability of $b_\alpha \in G_\alpha$ in the projection μ_α of the measure μ would be not less than $1/s_\alpha$. For b_1 take the element of G_1 having the largest probability in μ_1 . Since $|G_1| = s_1$, we have

$$\mu_1\{b_1\} \geq 1/s_1.$$

Suppose that the first k elements

$$b_1, b_2, \dots, b_k$$

have already been defined, for which

$$\mu_i(b_i) \geq 1/s_i, \quad \varphi_i(b_{i+1}) = b_i.$$

Define the element b_{k+1} . To this end, consider the complete inverse image of the element b_k under the homomorphism φ_k . Denote it by B_k . B_k contains

$$s_{k+1}/s_k = r_k$$

elements. By the construction of the projections μ we have

$$\mu_{k+1}(\{B_k\}) = \mu_k(b_k) \geq 1/s_k.$$

Consequently, in B_k there is an element b_{k+1} for which

$$\mu_{k+1}(b_{k+1}) \geq 1/s_k r_k = 1/s_{k+1}.$$

Thus, the required element

$$b = (b_1, b_2, \dots)$$

exists.

Denote

$$b^{-1} = (b_1^{-1}, b_2^{-1}, \dots)$$

by $c(\mu)$, and from the class of translates containing the measure μ , choose the representative equal to

$$\mu c(\mu).$$

Now take one arbitrary measure μ from each class of translates and form A , consisting of the collection of measures

$$\{\mu c(\mu)\}.$$

To prove Theorem 2 it remains to show that the set of measures A is complete.

Let

$$\{\mu_m c(\mu_m)\}$$

be an arbitrary sequence of measures from A . In order that

$$\prod \mu_m c(\mu_m)$$

converge, it is enough that the products of the Fourier coefficients of these measures converge for any finite-dimensional representations φ of the group G . But this convergence will follow from the convergence of the product of measures

$$\prod \mu_m(g),$$

where $\mu_m(g)$ are measures on the group

$$g = \varphi(G_\omega),$$

induced by the measures $\mu_m c(\mu_m)$ under the mapping φ .

Let N be the kernel of the representation φ . The factor group

$$g = G/N$$

is finite, for otherwise the group G_ω would not be zero-dimensional. Therefore, if the probabilities of the identity in the measures $\mu_m c(\mu_m)$ are uniformly bounded below by some positive number ε , then, by virtue of ⁽²⁾, the product of measures

$$\prod \mu_m(g)$$

converges. But the probability of the identity in the measure $\mu_m(g)$ is obviously equal to the probability of the subgroup N in the measure $\mu_m c(\mu_m)$.

It can be shown that the group N contains the set $[e_n]$ for some n , where e_n is the identity of G_n . The probability of $[e_n]$ in the measure $\mu_m c(\mu_m)$, by the definition of the projection of measures, is equal to the probability of the identity in the projection $\mu_m c(\mu_m)$ onto G_n . But this probability, in view of $c(\mu) = (b_1^{-1}, b_2^{-1}, \dots)$, is equal to the probability of the set $[b_n]$ in μ_m , which by construction is not less than $1/s_n$. But $[e_n] \subseteq N$. Therefore the probability of the subgroup N in $\mu_m c(\mu_m)$ is not less than $1/s_n$ for all $m = 1, 2, \dots, \infty$. The theorem is proved.

Theorem 3. *For distributions on a compact group G of positive dimension, it is impossible to form a complete set of distributions from representatives of each class of shifts.*

We shall first prove two auxiliary propositions.

Lemma 4. *There exists a sequence of measures on the circle $\mu_1, \mu_2, \dots$ such that no representatives of the shift classes containing the measures $\mu_1, \mu_2, \dots$ form a complete set.*

Proof. Let μ_n be a distribution on the circle in which, with probability $1/2$, the element $\exp\{i2\pi/f(n)\}$ is taken, and with probability $1/2$ the identity element of the group is taken. If $f(n)$ satisfies the conditions: $f(n)$ are even positive integers, $\sum 1/f(n) = \infty$, $\sum 1/f(n)^2 < \infty$, then the measures μ_n satisfy the condition of the lemma. The proof of this is based on the fact that a representative of the shift class containing the measure μ_n must be of the form $\exp\{i\frac{2\pi}{f(n)}m(n)\}\mu_n$, where $m(n)$ is a positive integer not exceeding $f(n)$. Otherwise the degree of this representative would not converge, which would violate the completeness of the set of representatives.

Lemma 5. *Let g be a commutative subgroup of G , and let A be a complete set of measures consisting of representatives of all shift classes. Then for any countable sequence of measures $\mu_1, \mu_2, \dots$ with supports in g , there exist elements $a_1, a_2, \dots$ belonging to g , and an element d of G , such that the measures $d^{-1}(\mu_1 a_1)d, d^{-1}(\mu_2 a_2)d, \dots$ belong to A .*

Proof. For simplicity, we first consider the case of two measures μ_1 and μ_2 . From these measures construct the sequence of distributions $\mu_1 b_1, b_1^{-1} \mu_2 b_2, \dots, b_{2n-1}^{-1} \mu_2 b_{2n}, \dots$, where b_n and b_{n+1} are connected by the relations $b_{2n} = c(b_{2n-1}^{-1} \mu_2)$ and $b_{2n-1} = c(b_{2(n-1)}^{-1} \mu_1)$.

The function $c(\cdot)$ is defined on G so that the measure $\mu c(\mu)$ is a representative of the shift class containing the measure μ . Therefore the products of the first $2n-1$ and $2n$ terms of this sequence, equal respectively to $(\mu_1 \mu_2)^{n-1} \mu_1 b_{2n-1}$ and $(\mu_1 \mu_2)^n b_{2n}$, have the same limits as $n \rightarrow \infty$.

By the convergence principle, the powers $(\mu_1 \mu_2)^n$ will converge to the invariant measure on the group g_1 , $g_1 \subseteq g$, if they are shifted by the corresponding

elements $\alpha_n, \alpha_n \in g$. Since $(\mu_1\mu_2)^n\alpha_n\alpha_n^{-1}b_{2n}$ converges, it follows that $\alpha_n^{-1}b_{2n}$ must be equal to $g_1^{(n)}d_n^{(1)}$, where $g_1^{(n)} \subseteq g_1 \subseteq g$, and the element $d_n^{(1)}$ tends to some $d \in G$. Similarly, $(\mu_1\mu_2)^{n-1}\mu_1b_{2n-1} = \mu_1(\mu_1\mu_2)^{n-1}\alpha_{n-1}\alpha_{n-1}^{-1}b_{2n-1}$ (by the commutativity of g) converges to the same limit as $(\mu_1\mu_2)^nb_{2n}$. Since the support of μ_1 is contained in g , and g is commutative, we obtain $b_{2n-1} = g_2^{(n)}d_n^{(2)}$, where $g_2^{(n)} \subseteq g$, and $d_n^{(2)}$ tends to $d \in G$.

Taking now into account the form of the elements b_{2n-1} and b_{2n} and the compactness of the group G , one can find indices n_i such that $g_1^{(n_i)} \rightarrow q_1, g_2^{(n_i)} \rightarrow q_2, g_1^{(n_i-1)} \rightarrow q'_1, g_2^{(n_i-1)} \rightarrow q'_2$. It is clear that all these limits belong to g . Then $b_{2n_i}^{-1}\mu_2b_{2n_i} \rightarrow d^{-1}q_1^{-1}\mu_2q_1d$, which, by the commutativity of g , can be written in the form $d^{-1}\mu_2(q_1q_2^{-1})d$. Similarly, $b_{2(n_i-1)}^{-1}\mu_1b_{2(n_i-1)} \rightarrow d^{-1}(q'_2)^{-1}\mu_1(q'_1)d = d^{-1}\mu_1(q'_2)^{-1}q'_1d$. Denote $q_1q_2^{-1} = a_1 \in g, q'_1(q'_2)^{-1} = a_2 \in g$. By Theorem 1,

the limit measures $d^{-1}(\mu_1\alpha_1)d$ and $d^{-1}(\mu_2\alpha_2)d$ belong to A . Thus Lemma 5 is proved for two arbitrary measures μ_1 and μ_2 .

The case of a larger finite number of measures $\mu_1, \mu_2, \dots, \mu_n$ with supports belonging to g is, obviously, considered analogously.

Thus, for any initial measures $\mu_1, \mu_2, \dots, \mu_n$ of the sequence $\{\mu_m\}$, there will be found elements $\alpha_1^{(n)}, \dots, \alpha_n^{(n)} \in g$ and $d_n \in G$ such that the measures

$$d_n^{-1}(\mu_i\alpha_i^{(n)})d_n$$

belong to A for $i = 1, 2, \dots, n$.

Let now, for the sequence $\{d_n\}$, the element d of G be a limit point. Then for the pair $\{d_n, \alpha^{(n)}\}$ there exists a limit point $\{d, \alpha_i\}$, where $\alpha_i \in g$.

Since the measure $d_n^{-1}(\mu_i\alpha_i^{(n)})d_n \in A$, by virtue of Theorem 1 the measure $d^{-1}(\mu_i\alpha_i)d$ also belongs to A . The lemma is proved.

We now pass directly to the proof of Theorem 3.

Let the group G be of nonzero dimension. Then there exists a linear representation φ of the group G with kernel N such that G/N is a Lie group.

Let $\varphi(\mu)$ be the measure on G/N generated by the measure μ under the mapping φ . Suppose that Theorem 3 is not fulfilled for the group G , i.e., there exists a complete set A of representatives of each class of shifts on G . But from the completeness of A follows the completeness of $\varphi(A)$, which also consists of representatives of all classes, but now on the group G/N . Thus there exists a Lie group $G_0 = G/N$ for which Theorem 3 does not hold. In every Lie group there is a subgroup that is a circle. Denote one of such subgroups G_0 by g . On g consider measures $\mu_1, \mu_2, \dots$ satisfying the conditions of Lemma 4. According to Lemma 5, there exist elements $\alpha_1, \alpha_2, \dots \in g$ and $d \in G_0$ such that the measures $d^{-1}(\mu_i\alpha_i)d, i = 1, 2, \dots, \infty$, belong to A . The collection of measures $d^{-1}(\mu_i\alpha_i)d, i = 1, 2, \dots, \infty$, must be complete as a subset of the complete set A . But then,

by virtue of Lemma 3, the sequence $\{\mu_i \alpha_i\}$ is complete on the circle g , which contradicts Lemma 4. The theorem is proved.

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Note: Figure translations are in progress. See original paper for figures.

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