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Abstract

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MATHEMATICS

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ON OPERATORS IN HÖLDER SPACES

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Everywhere below, $\varphi(\delta)$ ($0 \leq \delta \leq 1$) denotes a continuous increasing concave function, $\varphi(0) = 0$. By H_φ we denote the space of functions $u(x)$ continuous on $[0, 1]$, for which the norm

$$\|u\|_\varphi = \max\{\|u\|_C, \sup_{0 < \delta \leq 1} \omega(u; \delta)/\varphi(\delta)\},$$

is finite, where $\omega(u; \delta)$ is the modulus of continuity of the function $u(x)$. H_φ is called a generalized Hölder space. The classical Hölder spaces H_α are a special case of the spaces H_φ when $\varphi(\delta) = \delta^\alpha$. By H_φ^0 we denote the subspace of those functions from H_φ for which $\omega(u; \delta) = o[\varphi(\delta)]$.

1. Denote by K_φ the cone of nonnegative functions in H_φ . This cone is, obviously, solid and minihedral. It is easy to show that the cone K_φ is not normal (all definitions from the theory of Banach spaces with a cone used here may be found in ⁽¹⁾).

Theorem 1. *A positive ($AK_\varphi \subset K_{\varphi_1}$) linear (additive and homogeneous) operator A , acting from H_φ into H_{φ_1} , is continuous.*

In the proof of this assertion, the theorem of I. A. Bakhtin, M. A. Krasnosel'skii, and V. Ya. Stetsenko is used (Theorem 2.3 from ⁽¹⁾). Although the cone K_φ is not normal, the application of this theorem becomes possible thanks to the following simple but useful assertion.

Lemma. *Let a linear operator A defined on a Banach space E_1 act into a Banach space E_2 . Let the space E_2 be continuously embedded in a Hausdorff linear topological space $\mathcal{E}$, and suppose that A , as an operator from E_1 into $\mathcal{E}$, is continuous. Then A is continuous as an operator from E_1 into E_2 .*

For integral operators in Theorem 1 one can dispense with the positivity condition.

Theorem 2. *If the linear integral operator*

$$Au(x) = \int_0^1 K(x, y)u(y) dy \quad (1)$$

acts from H_φ into H_{φ_1} , then it is continuous.

We give some conditions under which a linear integral operator acts from a Banach space E into the Hölder space H_φ . Let M_φ be the Marcinkiewicz space (see, for example, ⁽²⁻⁴⁾), generated by the concave function $\varphi(\delta)$; M_φ^0 is the subspace of functions from M_φ with absolutely continuous norm.

Theorem 3. *If the function $K(z)$ belongs to M_φ , then the convolution operator*

$$Bu(x) = \int_0^x K(x-y)u(y) dy \quad (2)$$

acts into H_φ and is continuous. If $K(z) \in H_\varphi^0$, then this operator is continuous in H_φ^0 .

A well-known theorem of L. V. Kantorovich ⁽⁵⁾ on conditions under which the integral operator (1) acts from $L_p(\Omega)$ into $H_\alpha(\Omega)$ admits a generalization to the case of an arbitrary ideal space $E(\Omega)$ ⁽⁴⁾ and the space $H_\varphi(\Omega)$, respectively. For simplicity we restrict ourselves to the formulation of the corresponding assertion in the one-dimensional case. Below, E' denotes the space dual to the ideal space E .

Theorem 4. *Let the kernel $K(x, y)$ of the integral operator (1) be differentiable with respect to x for $x \neq y$. Suppose that the functions $K(x, y)/\varphi(|x-y|)$, $K'_x(x, y)|x-y|/\varphi(|x-y|)$, as functions of y , belong to E' for each x , and*

$$\|K(x, y)/\varphi(|x-y|)\|_{E'} \leq M; \quad \|K'_x(x, y)|x-y|/\varphi(|x-y|)\|_{E'} \leq N.$$

Then the operator (1) acts from E into H_φ and is continuous.

2. In the study of operators an important role is played by multiplicative inequalities connecting norms in different spaces (see, for example, ^(6,7)). For the spaces H_α such inequalities were first indicated, apparently, by Kh. Sh. Mukhtarov ⁽⁸⁾. Below we give an analogue of a multiplicative inequality connecting the norm of a function $u(x)$ from H_φ with the norms of this function in the space C and in an arbitrary symmetric space E ⁽³⁾. Below, $\psi(\delta)$ denotes the fundamental function of the symmetric space E .

Theorem 5. *The inequality*

$$\|u\|_C \leq F(\|u\|_\varphi, \|u\|_E) \quad (u(x) \in H_\varphi), \quad (3)$$

holds, where

$$F(t, s) = \min_{0 < \delta \leq 1/2} \left[t\varphi(\delta) + \frac{s}{\psi(\delta)} \right] \quad (0 \leq s \leq t\psi(1)).$$

The function $F(t, s)$ has the following properties:

1) $F(t, s)$ is jointly continuous in the variables and is a nondecreasing concave function in each of the variables.

2) $F(t, 0) = 0$ for all $t \geq 0$.

3) $F(t, kt) = c(k)t$.

For the case where $\psi(\delta) = [\varphi(\delta)]^r$ ($r > 0$), inequality (3) has the form

$$\|u\|_C \leq c(r) \|u\|_\varphi^{r/(1+r)} \|u\|_E^{1/(1+r)} \quad (u(x) \in H_\varphi).$$

In particular, for $\psi(\delta) = \varphi(\delta)$,

$$\|u\|_C \leq c \sqrt{\|u\|_\varphi \|u\|_E}.$$

It follows from inequality (3), for example, that every sequence of functions bounded in H_φ and converging in measure converges uniformly.

3. Denote by f the superposition operator generated by a certain function $g(x, u)$:

$$fu(x) = g[x, u(x)]. \quad (4)$$

As an operator acting in various spaces of summable functions, it has been studied by many authors (see, for example, ^(4, 6), where a bibliography is given). As an operator acting in Hölder spaces, the operator (4) has in general been studied hardly at all. Some results for the simplest case, when the function $g(x, u) \equiv g(u)$ does not depend on x , were obtained by A. A. Babaev (for the spaces H_α) and Kh. Sh. Mukhtarov (for the spaces H_φ). A more complete investigation of such a simplest operator was carried out by one of the authors of the present note in ⁽⁹⁾. Here we formulate two theorems on the operator (4) in the general case.

Denote by $\mathfrak{M}$ the class of increasing functions $a(t)$, continuous on $[0, \infty)$, satisfying the conditions:

- a) $\alpha(t) > 0$ for $t > 0$, $\alpha(0) = 0$;
- b) $\lim_{\delta \rightarrow 0} \alpha[\varphi(\delta)]/\varphi_1(\delta) > 0$.

By $\mathfrak{M}_1$ we shall denote the class of such functions $\alpha(t)$ which, instead of condition b), satisfy the condition

b') $\lim_{\delta \rightarrow 0} \alpha[l\varphi(\delta)]/\varphi_1(\delta) < \infty$ for every $l > 0$.

Theorem 6. In order that the operator (4) act from H_φ to H_{φ_1} and be bounded, it is necessary that, for every $r > 0$ and every function $\alpha(t)$ from $\mathfrak{M}$, the function $g(x, u)$ satisfy the condition

$$|g(x_1, u_1) - g(x_2, u_2)| \leq M(r, \alpha) [\varphi_1(|x_1 - x_2|) + \alpha(|u_1 - u_2|)]$$

$$(x_1, x_2 \in [0, 1], \quad u_1, u_2 \in [-r, r]), \quad (5)$$

where the constant $M(r, \alpha)$ does not depend on x, u .

If, for some function $\alpha(t)$ from $\mathfrak{M}_1$, (5) is satisfied, then the operator (4) acts from H_φ to H_{φ_1} and is bounded.

A consequence of this theorem is

Theorem 7. In order that the operator (4) act in H_φ and be bounded, it is necessary and sufficient that, for every $r > 0$, the function $g(x, u)$ satisfy the condition

$$|g(x_1, u_1) - g(x_2, u_2)| \leq M(r) [\varphi(|x_1 - x_2|) + |u_1 - u_2|]$$

$$(x_1, x_2 \in [0, 1], \quad u_1, u_2 \in [-r, r]).$$

The sufficiency of the conditions of these theorems is obvious. It has been used by many authors.

In [9], for the operator (4) generated by a function $g(u)$ independent of x , analogous theorems were proved without the additional assumption of boundedness of the operator f . In the general case this assumption is essential, since one can construct unbounded operators (4) acting in H_φ .

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