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Abstract

Full Text

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MATHEMATICS

V. S. RETAKH

ON SUBSPACES OF A COUNTABLE INDUCTIVE LIMIT

(Presented by Academician A. N. Kolmogorov, 20 III 1970)

Let $E = \lim_{\rightarrow} E_n$ be an inductive limit of locally convex spaces and let H be a subspace of E (in the topology induced from E). We shall say that H has property (T_0) if $H' = (\lim_{\rightarrow} (E_n \cap H))'$, and property (T) if $H = \lim_{\rightarrow} (E_n \cap H)$. Obviously, (T) implies (T_0) . We shall give necessary and sufficient conditions for the fulfillment of properties (T_0) and (T) under certain assumptions concerning E .

1. Let $G = \lim_{\rightarrow} G_n$ be an inductive limit of locally convex spaces. We shall say that G has property (M_0) if all G_n are of countable type* and contain an absolutely convex neighborhood of zero U_n such that $U_n \subset U_{n+1}$ and

$$\forall i \exists j : \forall k > j \forall f \in G'_j \forall \varepsilon > 0 \exists g \in G'_k : |f(x) - g(x)| < \varepsilon \forall x \in U_i. \quad (*)$$

If condition $(*)$ is replaced by the following: $\forall i \exists j : \forall k > j$ the topologies induced on U_i from G_j and G_k coincide, then we shall say that G has property (M) . (M) implies (M_0) . If $E = \lim_{\rightarrow} E_n$ is a strict inductive limit of spaces of countable type, then E has property (M) .

Theorem 1. If

$$E/H = \lim_{\rightarrow} E_n/E_n \cap H$$

has property (M_0) , then H has property (T_0) . If E has property (M_0) , or E is a strict inductive limit** and all $E_n/E_n \cap H$ are of countable type, then the converse assertion is also true.

Theorem 2. If E/H has property (M) , then H has property (T) . If E has property (M) , or E is a strict inductive limit** and all $E_n/E_n \cap H$ are of countable type, then the converse assertion is also true.

Remark. According to $(^7)$, if E is a strict inductive limit of Fréchet-Schwartz spaces, then for a closed H property (T_0) is equivalent to property (T) .

For the proof of Theorems 1 and 2 one must use the scheme of the proofs of Theorem 2 from ⁽³⁾ and Theorem 1' from ⁽⁴⁾, and Theorem 3 below.

We shall give the example, constructed by V. P. Palamodov and the author, of a closed subspace H of the space $D((-1, 2))$ not having property (T_0) . It differs from the example in ⁽⁶⁾ by its simplicity and by the fact that H is the image of an operator from the space D .

Let $\varphi \in D(R)$, with $\text{supp } \varphi = [1, 2]$. Define a mapping

$$Q : D((-1, 0)) \rightarrow D((-1, 2))$$

by the formula $Q(f) = f + f * \varphi$. Put $H = \text{Im } Q$,

$$E_n = D([-1 + 2^{-n}, 2 - 2^{-n}]), \quad H_n = E_n \cap H.$$

It is easy to see that Q is injective and H is closed. Suppose that H has property (T_0) . Since E is a strict-

* I.e., they possess a countable fundamental system of neighborhoods of zero.

** In this case it is not assumed that all E_n are of countable type.

inductive limit, then by Theorem 1 there exist a neighborhood of zero V_3 in E_3 and $j > 3$ such that

$$\forall u \in E'_j, \quad u|_{H_j} = 0 \quad \forall \varepsilon > 0 \exists v \in E'_{j+1}, \quad v|_{H_{j+1}} = 0 : |u(x) - v(x)| < \varepsilon \quad \forall x \in V_3.$$

Obviously $\delta_{-2^{-j}}|_{H_j} = 0$. Let $v \in E'_{j+1}$ and $v|_{H_{j+1}} = 0$. The latter means that $v(f + f * \varphi) = 0, \forall f \in D([-1 + 2^{-j-1}, -2^{-j-1}])$, whence it follows that v is infinitely differentiable on $[-1 + 2^{-j-1}, 2^{-j-1}]$. But $\delta_{-2^{-j}}$ cannot be approximated by infinitely differentiable functions on V_3 . We have arrived at a contradiction; consequently, H does not have property (T_0) .

Let us note that it follows from this that Q^{-1} is discontinuous.

From Theorem 2, with the aid of the 3×3 -lemma for the category of locally convex spaces ⁽²⁾, there follows the main result of ⁽⁵⁾. However, instead of Theorem 2 it suffices to use the weaker Theorem 1 from ⁽³⁾.

- II. Let L be the category of vector spaces over the field $\mathbf{C}$ or $\mathbf{R}$. By $\tilde{L}$ we shall denote the category of inverse spectra over L with the natural series $\mathbf{N}$, ordered increasingly, as the common set of indices.

V. P. Palamodov in ⁽¹⁾ studied the derived functors of the left exact functor $\text{Pro} : \tilde{L} \rightarrow L$, which assigns to each inverse spectrum $\mathcal{X} = \{X_p, \alpha_p^q : X_q \rightarrow X_p\}$ from $\tilde{L}$ its projective limit. He found a necessary and sufficient condition for $\text{Pro}^1 \mathcal{X}$ to vanish in the case when the spaces X_p ($p \in \mathbf{N}$) can be endowed with the topologies of Fréchet spaces so that the mappings α_p^q are continuous.

We shall prove an analogous theorem for the dual, in a certain sense, case.

Definition. An absolutely convex set B in a vector space E will be called a Banach disk if the space E_B , spanned by B and endowed, as a seminorm, with its Minkowski functional, is Banach.

Theorem 3. *Let in each space X_p of the inverse spectrum*

$$\mathcal{X} = \{X_p, \alpha_p^q\}$$

from $\tilde{L}$ there exist a sequence of Banach disks

$$(B_p^k)_{k \in \mathbf{N}}$$

such that

$$\bigcup_{k=1}^{\infty} B_p^k = X_p$$

and

$$[(\alpha_p^{p+1})^{-1}(B_p^k)] \cap B_{p+1}^r$$

is a Banach disk for arbitrary p, k, r . Then $\text{Pro}^1 \mathcal{X} = 0$ if and only if in each X_p there exists a Banach disk B_p such that

$$\alpha_p^{p+1}(B_{p+1}) \subset B_p$$

and

$$\forall i \exists j : \alpha_i^j(X_j) \subset B_i + \alpha_i(\text{Pro } \mathcal{X}).$$

For the proof we shall need a lemma which in essence belongs to Banach.

Lemma. *Let X, Y be vector spaces and $f : X \rightarrow Y$ a linear mapping. Suppose that the sequences of absolutely convex sets $(U_n)_{n \in \mathbf{N}}$ and $(V_n)_{n \in \mathbf{N}}$ in X and Y , respectively, form fundamental systems of neighborhoods of zero for the topologies of separable additive groups. If the group X is complete, the graph of f is closed, and $f(U_n)$ is, for every n , of second category in Y , then f is open.*

Proof of Theorem 3. Sufficiency follows from Theorem 11.2 in ⁽¹⁾. We shall prove necessity.

Put

$$\Pi = \prod_{p=1}^{\infty} X_p.$$

According to ⁽¹⁾, if $\text{Pro}^1 \mathcal{X} = 0$, then the mapping

$$\pi : \Pi \rightarrow \Pi$$

defined by the formula

$$\pi(x_1, x_2, x_3, \dots) = (x_1 - \alpha_1^2(x_2), x_2 - \alpha_2^3(x_3), \dots) \quad (**)$$

is epimorphic. Introduce in Π the topology φ of an additive group with a fundamental system of neighborhoods of zero $(V_n)_{n \in \mathbb{N}}$, where

$$V_n = 0 \times \dots \times 0 \times X_n \times X_{n+1} \times \dots$$

(the 0 occurring $n - 1$ times). Since (Π, φ) is complete and π is epimorphic, for each p -

there exists B_n^k such that $\pi(B_1^{k_1} \times \dots \times B_m^{k_m} \times X_{m+1} \times X_{m+2} \times \dots)$ is of the second category in (Π, φ) , for every m . Introduce on Π the topology τ of a group with a fundamental system of neighborhoods of zero $(U_n)_{n \in \mathbb{N}}$, where

$$U_n = 2^{-n} B_1^{k_1} \times \dots \times 2^{-n} B_n^{k_n} \times X_{n+1} \times X_{n+2} \times \dots$$

By the lemma, $\pi : (\Pi, \tau) \rightarrow (\Pi, \varphi)$ is open. Hence $\forall i \exists j :$

$$\pi(B_1^{k_1} \times \dots \times B_i^{k_i} \times X_{i+1} \times X_{i+2} \times \dots) \subset (0 \times \dots \times 0 \times X_j \times 0 \times \dots).$$

To complete the proof one must use formula (**).

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