

ON THE NON-SELF-ADJOINT OPERATOR $\backslash(-y''+p(x)y\backslash)$ ON THE AXIS $\backslash((-\infty,\infty)\backslash)$

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Abstract

Full Text

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MATHEMATICS

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ON THE NON-SELF-ADJOINT OPERATOR $-y'' + p(x)y$ ON THE AXIS $(-\infty, \infty)$

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We consider the differential equation

$$-y'' + p(x)y = \lambda y, \quad (1)$$

where $p(x) = q(x) + ir(x)$ is a complex-valued function of the real argument x , and λ is a complex parameter. Associated with equation (1) is the non-self-adjoint differential operator acting in $L^2(-\infty, \infty)$,

$$Ly = -y'' + p(x)y,$$

whose domain $D(L)$ consists of functions $y(x) \in L^2(-\infty, \infty)$ that are absolutely continuous on every finite interval together with their derivative and such that $-y'' + p(x)y \in L^2(-\infty, \infty)$.

In the present paper, with the aid of an auxiliary system of nonlinear first-order differential equations, we investigate the question of the existence of nontrivial solutions of equation (1) belonging to $L^2(-\infty, \infty)$, and give conditions imposed on $p(x)$ under which the operator L has a completely continuous resolvent and, consequently, a discrete spectrum. The results presented in the paper are adjacent to certain results of M. A. Naimark ⁽¹⁾ and V. B. Lidskii ⁽²⁾.

Consider the equation

$$-y'' + p(x)y = 0 \quad (2)$$

and suppose that for $x \in [a, b]$ the function $p(x)$ can be represented in the form

$$p(x) = \rho(x)e^{i\varphi_0(x)},$$

where $\rho(x)$ and $\varphi_0(x)$ are continuous functions, and

$$\rho(x) \geq \rho_0 > 0. \quad (3)$$

Make in (2) the substitution

$$y' = y \operatorname{ctg} \theta(x) e^{i\varphi(x)}.$$

This substitution is an analogue of Prüfer's substitution ⁽³⁾, often used in the study of the self-adjoint case. For the functions $\theta(x)$ and $\varphi(x)$ we obtain the system of nonlinear equations

$$\begin{aligned} \theta' &= \cos^2 \theta \cos \varphi - \sin^2 \theta \rho(x) \cos(\varphi - \varphi_0(x)), \\ \varphi' &= -\operatorname{ctg} \theta \sin \varphi - \operatorname{tg} \theta \rho(x) \sin(\varphi - \varphi_0(x)). \end{aligned} \quad (4)$$

Theorem 1. *To every nontrivial solution $y(x)$ of equation (2) there corresponds a pair of real continuously differentiable functions $\theta(x)$ and $\varphi(x)$ on the interval $[a, b]$ such that*

$$y'(x)/y(x) = \operatorname{ctg} \theta(x) e^{i\varphi(x)}.$$

The functions $\theta(x)$ and $\varphi(x)$ satisfy system (4) and, moreover, possess the following properties:

- 1) $\theta(x)$ can cross the lines $\theta = k\pi$ from below upward only at those points where $\varphi(x) = 2m\pi$, and from above downward only at those points where $\varphi(x) = (2m+1)\pi$.
- 2) $\theta(x)$ can cross the lines $\theta = \pi/2 + k\pi$ from below upward only at those points where $\varphi(x) = \varphi_0(x) + (2m+1)\pi$, and from above downward only at those points where $\varphi(x) = \varphi_0(x) + 2m\pi$.

The functions $\theta(x)$ and $\varphi(x)$ are determined uniquely up to the transformations

$$\theta = \theta + n\pi; \quad \varphi = \varphi + 2n\pi; \quad \theta = -\theta, \quad \varphi = \varphi + \pi.$$

Suppose that the function $p(x) = \rho(x)e^{i\varphi_0(x)}$ is such that, for $x \in [a, b]$, its values lie in an open angle whose closure does not contain the negative real semiaxis, i.e., for $x \in [a, b]$,

$$\gamma < \arg \varphi_0(x) < \delta, \quad (5)$$

where $-\pi < \gamma < 0$, $0 < \delta < \pi$.

Lemma 1. Let the function $p(x)$ satisfy conditions (3) and (5) on the segment $[a, b]$, with $\delta - \gamma \leq \pi$. Then, if the functions $\theta(x)$ and $\varphi(x)$ are solutions of system (4) such that

$$0 < \theta(a) < \pi/2, \quad \gamma < \varphi(a) < \delta,$$

then for all $x \in [a, b]$ the relations

$$0 < \theta(x) < \pi/2, \quad \gamma < \varphi(x) < \delta$$

hold.

If $p(x)$ is continuously differentiable for sufficiently large values of $|x|$, satisfies conditions (3) and (5) on the interval $(-\infty, \infty)$, and

$$\lim_{|x| \rightarrow \infty} p'(x)/p^{3/2}(x) = 0, \quad (6)$$

then equation (2) has solutions $\chi(x)$ and $\psi(x)$, unique up to constant factors, belonging respectively to $L^2(-\infty, 0)$ and $L^2(0, \infty)$.

Moreover,

$$\chi'(x)/\chi(x) = \sqrt{p(x)}(1 + o(1)), \quad x \rightarrow -\infty, \quad (7)$$

$$\psi'(x)/\psi(x) = -\sqrt{p(x)}(1 + o(1)), \quad x \rightarrow +\infty. \quad (8)$$

The value of $\sqrt{p(x)}$ is chosen in the right half-plane.

Using Lemma 1 and the asymptotic formulas (7) and (8), one can prove the following lemma.

Lemma 2. Let the function $p(x) = \rho(x)e^{i\varphi_0(x)}$ be continuously differentiable on the interval $(-\infty, \infty)$, satisfy conditions (3), (5), and (6) on this interval, and let, moreover, for $x \in (-\infty, \infty)$,

$$|\varphi'_0(x)|/\sqrt{\rho(x)} \leq d,$$

where $d = \min(|\sin \gamma|, \sin \delta, \cos \gamma/2, \cos \delta/2)$. Then the solutions $\chi(x)$ and $\psi(x)$ are linearly independent, and equation (2) has no nontrivial solutions belonging to $L^2(-\infty, \infty)$.

Put

$$V_c^{\gamma, \delta} = \{z : \gamma < \arg(z - c) < \delta\}.$$

From Lemma 2 it follows:

Theorem 2. Let the function $p(x)$ in the equation

$$-y'' + p(x)y = \lambda y \quad (9)$$

satisfies condition (6), and suppose there exists a real c such that for $x \in (-\infty, \infty)$

$$p(x) \in V_c^{\gamma, \delta},$$

where $-\pi < \gamma < 0$, $0 < \delta < \pi$. Then there exists a real number λ_0 such that, for real $\lambda \leq \lambda_0$, equation (9) has no nontrivial solutions belonging to $L^2(-\infty, \infty)$.

When the conditions of Theorem 2 are fulfilled, on the real half-axis $\lambda \leq \lambda_0$ there are no points of the discrete spectrum of the operator L . The following theorem gives a sufficient condition for discreteness of the spectrum of the operator L .

Theorem 3. Suppose the function $p(x) = q(x) + ir(x)$ satisfies condition (6), and suppose there exist real constants k^- and k^+ such that

$$\lim_{x \rightarrow -\infty} (q(x) + k^- r(x)) = +\infty,$$

$$\lim_{x \rightarrow +\infty} (q(x) + k^+ r(x)) = +\infty.$$

Then the operator

$$Ly = -y'' + p(x)y$$

has a completely continuous resolvent and, consequently, a purely discrete spectrum.

We note that under the assumptions of Theorem 3 the set of values of the quadratic functional (Ly, y) may, in contrast to the cases considered in (2), fill the entire plane.

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Note: Figure translations are in progress. See original paper for figures.

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