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Abstract

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MATHEMATICS

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ON THE SUMMABILITY OF POSITIVE SOLUTIONS OF ELLIPTIC EQUATIONS OF ARBITRARY ORDER

(Presented by Academician S. L. Sobolev on 20 I 1970)

Here we shall set forth the results of a study of positive generalized, in the sense of S. L. Sobolev, (weak) solutions of elliptic equations in a bounded domain Ω with smooth boundary S , i.e., functions $u(x)$, summable in every subdomain Ω_1 of the domain Ω , satisfying the integral identity

$$\iint_{\Omega_1} P^* \Phi(x) u(x) dx \equiv \iint_{\Omega_1} \sum_{|k| \leq m} a_k(x) D_x^k \Phi(x) u(x) dx = \iint_{\Omega_1} \Phi(x) f(x) dx; \quad (1)$$

$\Phi(x)$ is any test function having m continuous derivatives in $\overline{\Omega_1}$ and vanishing, together with its derivatives up to order $m - 1$, on the boundary $\partial\Omega_1$ of the domain Ω_1 .

Denote by $\rho(x, S)$ the distance from the point $x \in \Omega$ to S , and by $L_1^{(t)}(\Omega)$ the collection of functions, summable in every subdomain of the domain Ω , for which the norm

$$\|u\|_{\Omega; t} = \iint_{\Omega} \rho(x, S)^t |u(x)| dx$$

is bounded.

The main result of the present work is that every positive function (having a negative component from $L_1^{(\gamma-m)}(\Omega)$) satisfying the integral identity (1) belongs to $L_1^{(\gamma-m)}(\Omega)$ with some $\gamma \geq m$ (γ , generally speaking, large), if $f(x) \in L_1^{(\gamma)}(\Omega)$. Algebraic conditions, of the type of strengthened ellipticity, are established for the membership of all positive weak solutions in the space $L_1^{(\gamma-m)}(\Omega)$ with a

prescribed $\gamma \geq m$. The investigation covers uniformly elliptic equations with bounded coefficients and equations that may degenerate on the boundary of the domain. Simple examples (ordinary differential equations) show that the algebraic conditions formulated are natural and sharp.

In the case of uniformly elliptic equations of arbitrary order with smooth coefficients, from our results and the recent work of Ya. A. Roitberg ⁽³⁾ there follows a theorem on the generalized integral representation of positive solutions. The methods developed here make it possible to substantially supplement the theorems on the behavior of solutions in unbounded domains presented in ⁽¹⁾. The present work continues the investigations of V. A. Kondrat'ev and one of the authors ⁽¹⁾.

1. Conditions.

Weak solutions of equations of the form

$$\sum_{|k| \leq m} (-1)^{|k|} D_x^k [a_k(x)u] = f(x). \quad (1')$$

are considered. In fact, we deal only with the integral identity (1). We introduce a number of conditions needed for the formulation of the results:

A_1 . $a_k(x)$ are measurable bounded functions, $|a_k(x)| \leq B$.

A_2 . The polynomial

$$P_0(x; \sigma) = \sum_{|k|=m} a_k(x) \sigma^k$$

is uniformly elliptic (u.e.) with u.e. constant δ_0 .

A_3 . The range of values of the polynomial $w = P_0(x; \sigma)$, where σ is any real vector, lies in the sector (cone) $K_{\varphi_1} = \{w; |\arg w| \leq \varphi_1 < \pi/2\}$ of the complex w -plane.

$A_4^{(\gamma)}$. There exists $\gamma \geq m$ for which the polynomial

$$Q_0^{(\gamma)}(x; \sigma) = \sum_{\mu=0}^m \prod_{s=0}^{\mu-1} \left(1 - \frac{s}{\gamma}\right) \frac{1}{\mu!} \frac{\partial^\mu}{\partial \sigma_n^\mu} P_0(x; \sigma) \Big|_{\sigma_n=0} \sigma_n^\mu$$

is uniformly elliptic with constant of uniform ellipticity $\delta_0^{(\gamma)}$.

$\tilde{A}_1$. The operator P^* has the form

$$P^*(x; D_x) \equiv \sum_{|k| \leq m} x_n^{|k|-m} b_k(x) D_x^k + \sum_{|k| \leq m-1} c_k(x) D_x^k \equiv$$

$$\equiv P_0^*(x; x_n^{-1}; D_x) + P_1^*(x; D_x); \quad P_0^*(x; \theta, D_x) = \sum_{|k| \leq m} \theta^{m-|k|} b_k(x) D_x^k,$$

$b_k(x)$, $c_k(x)x_n^{m-|k|}$ are measurable bounded functions; $|c_k(x)x_n^{m-|k|}| \leq B$; for sufficiently small x_n , $x_n \in (0, h)$, $|c_k(x)x_n^{m-|k|}| \leq q$, where q is sufficiently small.

$\tilde{A}_2$. The polynomial $\sum_{|k|=m} b_k(x)\sigma^k$ is uniformly elliptic with constant of uniform ellipticity δ_0^* .

$\tilde{A}_4^{(\gamma)}$. There exists $\gamma \geq m$ for which the polynomial

$$Q_0^{*(\gamma)}(x; y; \sigma) = \sum_{\mu=0}^m \prod_{s=1}^{\mu-1} \left(1 - \frac{s}{\gamma}\right) \frac{1}{\mu!} \frac{\partial^\mu}{\partial \sigma_n^\mu} P_0^*(x; \theta; \sigma) \Big|_{\sigma_n=0; \theta=\frac{\sigma_n}{\gamma}y}, \quad y \in [0, 1],$$

is uniformly elliptic with constant of uniform ellipticity $\delta_0^{*(\gamma)}$.

$\tilde{A}_3$. The range of values of the polynomial $w = \sum_{|k|=m} b_k(x)\sigma$ lies in the sector K_{φ_1} .

$A_3^{(\gamma)}(\tilde{A}_3^{(\gamma)})$. The range of values of the polynomial $w = Q_0^{(\gamma)}(x; \sigma)$ ($w = Q_0^{*(\gamma)}(x; y; \sigma)$) lies in the sector K_{φ_1} .

$A_5(\tilde{A}_5)$. $a_k(x)$ ($b_k(x)$) with $|k| = m$ are uniformly continuous.

The condition $A_4^{(\gamma)}(\tilde{A}_4^{(\gamma)})$ follows from $A_2(\tilde{A}_2)$ (the conditions of uniform ellipticity), and $A_3^{(\gamma)}(\tilde{A}_3^{(\gamma)})$ from $A_3(\tilde{A}_3)$ for sufficiently large γ .

The conditions $A_4^{(\gamma)}$, $\tilde{A}_4^{(\gamma)}$, $\tilde{A}_1$, $\tilde{A}_3^{(\gamma)}$, $A_3^{(\gamma)}$ are used in studying solutions defined in domains lying in the half-space $x_n > 0$ and adjacent to the hyperplane $x_n = 0$; we shall use the same conditions in the case of the half-space $x \cdot \nu > 0$, where ν is some vector (we shall denote them by the same letters with the index ν below). To write them down, new coordinates $\xi_1, \xi_2, \dots, \xi_n$ are introduced; the n -th coordinate axis is directed along the vector ν , and the remaining ones are placed in the plane perpendicular to ν so that they all form an orthogonal basis. In the new coordinates the conditions are written down—the role of x_n is played by ξ_n .

Denote by $K = K_{\varphi_2} = \{\zeta; |\arg \zeta| \leq \varphi_2 < \pi/2 - \varphi_1\}$; $u^+(x) = u(x)$ for $u(x) \in K$, $u^+(x) = 0$ if $u(x) \notin K$; $u(x) = u^+(x) - u^-(x)$.

$$\Pi_{a,h}^+ = \{x, x^2 = x_1^2 + x_2^2 + \dots + x_n^2 < a^2, x_n > h\}; \quad \Pi_{a,0}^+ = \Pi_a^+;$$

$$\Pi_{R,h}^{(\varepsilon)} = \{x, x'^2 = x_1^2 + \dots + x_{n-1}^2 < R^2, \varepsilon < x_n < h\}; \quad \Pi_{R,h}^{(0)} = \Pi_{R,h}.$$

In formulating the results, we shall everywhere assume that the constants entering into the formulation depend on the quantities entering into the assumed conditions and do not depend on the solutions under study.

2. Lemmas. Model problems.

Here the main assertions will be set forth that give the required estimates of solutions in the simplest domains: a half-ball, a cylinder of small height, and traces of a solution on a family of nearby surfaces. We emphasize especially that, in contrast to (1), here summability of the solutions up to the boundary is established rather than assumed.

boundary (in this item up to a piece of the hyperplane $x_n = 0$). The lemmas are of independent interest and have applications beyond those which we present in this paper.

Lemma 1 (on three half-balls). Suppose the conditions $A_1(\tilde{A}_1)$ hold in Π_2^+ ; $A_2(\tilde{A}_2)$, $A_3(\tilde{A}_3)$ (or $A_5(\tilde{A}_5)$) in Π_{2,ε_0}^+ ; $A_4^{(\gamma)}(\tilde{A}_4^{(\gamma)})$, $A_3^{(\gamma)}(\tilde{A}_3^{(\gamma)})$ in $\Pi_2^+ \cap \{x_n < \varepsilon_0\}$, $u(x)$ is a weak solution of equation (1') (of the equation with the adjoint operator from condition $\tilde{A}_1$). Then for any $a \in (1, 2)$ and $h \in (0, 1)$ there exists $\lambda_1 = \lambda_1(a, h)$ such that

$$\|u\|_{\Pi_a^+; \gamma-m} \leq \lambda_1 (\|u^+\|_{\Pi_{1,h}^+; \gamma-m} + \|u^-\|_{\Pi_2^+; \gamma-m} + \|f\|_{\Pi_2^+; \gamma}).$$

Lemma 2 (estimate of solutions in a circular cylinder of small height).

Suppose the conditions $A_1(\tilde{A}_1)$ hold in $R_{,1}$; $A_4^{(\gamma)}(\tilde{A}_4^{(\gamma)})$, $A_3^{(\gamma)}(\tilde{A}_3^{(\gamma)})$ in $R_{,\varepsilon_0}$, $u(x)$ is a weak solution of equation (1') (of the equation with the adjoint operator from condition $\tilde{A}_1$). Then for any $R_1 < R$ there exist positive constants $h_1 = h_1(R, R - R_1)$, $\lambda_2 = \lambda_2(R, R - R_1)$, such that

$$\|u\|_{R_{,1,h_1}; \gamma-m} \leq \lambda_2 (\|u\|_{R_{,1},0}^{h_1} + \|u^-\|_{R_{,1}; \gamma-m} + \|f\|_{R_{,1}; \gamma}).$$

The proofs of Lemmas 1, 2 are carried out by constructing families of admissible test functions $\Phi_\alpha(x)$ for elementary domains—lunes, half-lunes, cylinders of small height—for which $P^*\Phi_\alpha(x)$ have the positivity property (if the algebraic conditions $A_3^{(\gamma)}(\tilde{A}_3^{(\gamma)})$, $A_4^{(\gamma)}(\tilde{A}_4^{(\gamma)})$ are satisfied) and estimates, on the basis of the identity, $u^+(x)$. The arguments are very close to those set forth in ⁽¹⁾ (see also ⁽⁴⁾).

3. Positive solutions in bounded domains

Let Ω be a bounded domain with boundary $S = \partial\Omega$, belonging to the class $C^{(m)}(K_0)$ ⁽²⁾ (p. 341). Denote by K a constant (constants) characterizing the boundary S , by $\nu(x_0)$ the inward normal to S at the point x_0 , and by $\Omega_{\varepsilon_0} = \{x; x \in \Omega, \rho(x, S) \leq \varepsilon_0\}$ the ε_0 -neighborhood of the boundary S . Using the usual device of reducing the problem of studying a solution in a domain with smooth boundary to the study of model problems and then applying Lemmas 1 and 2, we arrive at the following basic theorem.

Theorem 1 (on summability of positive solutions). Suppose the conditions $A_1(\tilde{A}_1)$ hold in Ω , $A_2(\tilde{A}_2)$, $A_3(\tilde{A}_3)$ (or $A_5(\tilde{A}_5)$) in $\Omega \setminus \Omega_{\frac{1}{2}\varepsilon_0}$; $A_{4,\nu(x_0)}^{(\gamma)}$ ($\tilde{A}_{4,\nu(x_0)}^{(\gamma)}$), $A_{3,\nu(x_0)}^{(\gamma)}$ ($\tilde{A}_{3,\nu(x_0)}^{(\gamma)}$) (or the coefficients are real), x_0 is any point of S , in $\Omega \cap \{x, |x - x_s| < \varepsilon_0\}$, $u(x)$ is a weak solution of equation (1') (of the equation with the adjoint operator from condition $\tilde{A}_1$) in the domain Ω . Then for any subdomain Ω_1 of the domain Ω there exists a constant $C_1 = C_1(K, \rho(S, \partial\Omega_1))$ such that

$$\|u\|_{\Omega; \gamma-m} \leq C_1 (\|u\|_{\Omega_1; 0} + \|u^-\|_{\Omega; \gamma-m} + \|f\|_{\Omega; \gamma}).$$

For $\gamma = m$, Theorem 1 asserts the summability over the domain Ω of any weak positive solutions of the homogeneous equation.

We note that for degenerate equations it is necessary to write $\tilde{A}_{1,\nu(x_0)}$ for the adjoint operator only in a neighborhood of points $x_0 \in S$ (one may take a finite number of them), while in $\Omega \setminus \Omega_{\varepsilon_0}$ only ellipticity

$$\sum_{|k|=m} b_k(x) \sigma^k$$

and boundedness of the coefficients of the adjoint operator are assumed.

We shall give one more important consequence of Theorem 1 and of results of Ya. A. Roitberg ⁽³⁾ (all definitions and notation are taken from ⁽³⁾).

Theorem 2 (on a generalized integral representation). Suppose the conditions of Theorem 1 hold, and the coefficients p of the properly elliptic equation

$$Pu \equiv \sum_{|k| \leq m} (-1)^{|k|} D_x^k [a_k(x)u] = 0$$

are sufficiently smooth. Then any solution of it for which $\|u^-\|_{\Omega; \gamma-m} < +\infty$ is representable at every interior point x of the domain Ω by the formula

$$u(x) = - \sum_{j=1}^{m/2} \langle u_j, T_{m-j} \bar{R}(x, y) \rangle + \\ + \tilde{u}(x);$$

$R(x, y)$ is the Green's function of the Dirichlet problem for the equation $Pu = 0$ [7]; T_{m-j} are differential operators (with respect to y) of order $m - j$,

$$u_j \in W_p^{s_j}(S), \quad s_j < m - \gamma - (n - 1) + (n - 1)\frac{1}{p} - j;$$

$\langle u_j, \cdot \rangle$ is the value of a functional concentrated on S ; $\tilde{u}(x)$ is the solution of the homogeneous Dirichlet problem for the equation $Pu = 0$. By the assumptions made, $\tilde{u}(x)$ is a sufficiently smooth function in $\bar{\Omega}$.

4. Equation of the second order. Examples. For second-order equations with real coefficients

$$P_0(x, \sigma) \equiv \sum_{i,j=1}^n a_{ij}(x) \sigma_i \sigma_j \equiv A(x) \sigma \cdot \sigma; \quad P_0^*(x; \theta; \sigma) = \sum_{i,j=1}^n b_{ij}(x) \sigma_i \sigma_j + \\ + \theta \sum_{i=1}^n b_{i0}(x) \sigma_i + b_{00}(x) \theta^2.$$

In this case all the conditions reduce to boundedness of the coefficients and of the real parts of the polynomials

$$Q_0^{(\gamma)}(x; \sigma) \equiv A(x) \sigma \cdot \sigma - \frac{1}{\gamma} a_{nn}(x) \sigma_n^2; \\ Q_0^{*(\gamma)}(x; y; \sigma) \equiv \sum_{i,j=1}^n b_{ij}(x) \sigma_i \sigma_j + \frac{y}{\gamma} \sum_{i=1}^{n-1} b_{i0}(x) \sigma_i \sigma_n + \\ + \left[b_{nn}(x) \frac{y^2}{\gamma^2} + b_{n0}(x) \frac{y}{\gamma} - \frac{1}{\gamma} b_{nn}(x) \right] \sigma_n^2 \geq \delta^* > 0; \quad |\sigma| = 1, \quad y \in [0, 1].$$

The best condition $A_{4\nu}^{(\gamma)}$ for all ν is the validity of the inequality

$$\sup \frac{\lambda_1(x)}{\lambda_n(x)} < \frac{1 + \sqrt{1 - 1/\gamma}}{1 - \sqrt{1 - 1/\gamma}},$$

where $\lambda_1(x)$ is the largest, and $\lambda_n(x)$ the smallest, characteristic root of the symmetric matrix $A(x)$.

It is easy to give examples of equations for which condition $\tilde{A}_4^{(\gamma)}$ is not only sufficient, but also necessary for positive solutions to belong to the spaces

$$L_1^{(\gamma-m)}((0, 1)).$$

We restrict ourselves to the simplest. The equation

$$\begin{aligned} \frac{d^2 u}{dx_n^2} - \frac{d}{dx_n} [-(\alpha + 1)x_n^{-1}u] + (\alpha + 1)x_n^{-2}u &= 0; \quad Q_0^{*(\gamma)}(x_n; y; \sigma_n) = \\ &= [\gamma(\gamma - 1) - y\gamma(\alpha + 1) + y^2(\alpha + 1)] \sigma_n^2 / \gamma^2 \geq \delta^* \sigma_n^2 \quad \text{for } \gamma > 1 + \alpha, \end{aligned}$$

has the solution $x_n^{-\alpha}$, while the equation

$$\begin{aligned} \frac{d^2}{dx_n^2}(x_{nu}) - \frac{d}{dx_n} [-(1 + x_n^{-1})u] &= 0, \quad Q_0^{*(\gamma)}(x_n; y; \sigma_n) \equiv \\ &\equiv \left[x_n \left(1 - \frac{1}{\gamma} \right) - \frac{1}{\gamma} y \right] \sigma_n^2 \end{aligned}$$

has the solution e^{1/x_n} . The first shows that Theorem 1 on summability in the scale of spaces

$$L_1^{(\gamma-m)}(\Omega)$$

cannot be improved; the second shows that rejection of condition $\tilde{A}_4$ entails the appearance of positive solutions with non-power (exponential) singularities.

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Note: Figure translations are in progress. See original paper for figures.

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