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Abstract

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MATHEMATICS

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ON THE APPROXIMATE FINDING OF FIXED POINTS OF CONTINUOUS MAP- PINGS

(Presented by Academician L. V. Kantorovich, 16 VII 1969)

The note discusses methods for constructing approximations to fixed points under the conditions of the theorems of Brouwer, Kakutani, and Lefschetz-Hopf ^(1,4,5), using the apparatus of simplicial subdivisions. They may find application in nonlinear analysis, as well as in the solution of certain problems of mathematical economics ⁽²⁻⁴⁾.

1. Let $T \subset E^n$ be a closed connected set and let $f : T \rightarrow T$ be a continuous mapping. The set of fixed points of f , obviously, coincides with $\varphi^{-1}(0)$, where $\varphi(x) = f(x) - x$. To describe the construction of passage to a discrete mapping—numbering—we consider $Z_{n+1} = \{1, 2, \dots, n+1\}$ and a family of continuous vector-functions $v_i : T \rightarrow E^n$, $i \in Z_{n+1}$, such that for any $x \in T$ one has

$$\sum_{i=1}^{n+1} v_i(x) = 0, \quad (v_i(x), v_j(x)) \leq 0$$

for $i \neq j$, and the vectors $\{v_i(x)\}$, $1 \leq i \leq n$, are linearly independent. Further, let

$$M_i = \{x \in T \mid (\varphi(x), v_i(x)) \leq 0\}, \quad \mu(x) = \{i \in Z_{n+1} \mid x \in M_i\}.$$

Lemma 1.

$$\bigcap_{i=1}^{n+1} M_i = \varphi^{-1}(0), \quad \bigcup_{i=1}^{n+1} M_i = T$$

and, consequently, $\mu(x) \neq \emptyset$ for all $x \in T$.

Definition. A mapping $\nu : T \rightarrow Z_{n+1}$ such that $(\forall x) \nu(x) \in \mu(x)$ is called a **numbering**. A set $D \subset T$ is called **representative** for μ if

$$\bigcup_{x \in D} \mu(x) = Z_{n+1},$$

and a simplex σ is representative if its set of vertices is so. A **normal** ^(1,2) simplex in the numbering ν is an n -dimensional simplex σ^n (whose set of vertices is denoted by the same symbol) for which $\nu(\sigma^n) = Z_{n+1}$, and an $(n-1)$ -dimensional simplex σ^{n-1} is (Cj) -normal if

$$\nu(\sigma^{n-1}) = Z_{n+1} - \{j\}.$$

Obviously, a normal simplex is representative. Further, by Lemma 1 a one-point representative set is contained in $\varphi^{-1}(0)$. For the case when $T = (a_1, a_2, \dots, a_{n+1})$ is an n -dimensional simplex with vertices a_i , the classical numbering in the proof of Brouwer's theorem using Sperner's lemma is obtained with the aid of functions $v_i(x) = v_i^0$, where v_i^0 are the outer normals to the corresponding (not containing a_i) $(n-1)$ -dimensional faces of the original simplex, with

$$\sum v_i^0 = 0, \quad (v_i^0, v_j^0) \leq 0.$$

This presupposes a special realization of the simplex T in E^n . Below, in item 4, the inner normals of the corresponding faces are chosen as the v_i^0 , and it then turns out that $\mu(x) \supset Z_{n+1} - I = CI$ for all $x \in \Gamma(I) = (a_i)_{i \in I}$, where I is an arbitrary nonempty subset of Z_{n+1} , $\Gamma(I)$ is the face of the simplex T determined by the vertices a_i whose indices belong to I ; in other words,

$$\Gamma(I) \subset \bigcap_{i \in CI} M_i.$$

2. The estimate given here for the closeness of a representative simplex to a fixed point is essentially based on Lebesgue's lemma ^(1,5) on the existence, for a finite open covering of a metric compactum K , of a $\delta > 0$ (a Lebesgue number) such that every set $M \subset K$ with diameter less than δ is contained in one of the sets of the covering.

Let x^* be an isolated fixed point for f , and let a closed neighborhood U of this point contain no other fixed points. Put

$$P_\varepsilon = T \cap U \cap CS(x^*, \varepsilon), \quad Q = \{\varepsilon \mid S(x^*, \varepsilon) \subset U\},$$

$$\gamma(\varepsilon) = \min_{x \in P_\varepsilon} \max_{i \in Z_{n+1}} d(x, M_i), \quad \beta(p) = \sup\{\varepsilon \in Q \mid \gamma(\varepsilon) \leq p\},$$

where $S(x^*, \varepsilon)$ is the open sphere with center x^* and radius $\varepsilon > 0$, the symbol C , as usual, denotes the complement of the corresponding set, and $d(x, M_i)$ is the distance from x to M_i . Below, by the radius of a geometric simplex $\sigma^k = (a_1, a_2, \dots, a_{k+1})$ we mean

$$\rho = \min_{x \in \sigma} \max_{1 \leq i \leq k+1} d(x, a_i),$$

and the center of this simplex is considered to be a point z such that

$$\max_i d(z, a_i) = \rho.$$

Theorem 1. *Let the closed set U be a neighborhood of a fixed point x^* of the mapping $f : T \rightarrow T$, and let U contain no other fixed points of this mapping. Then, if σ is a representative simplex of radius ρ , whose vertices and center z lie in $T \cap U$, then $d(x^*, z) \leq \beta(\rho)$.*

Remark. In a number of cases, for example if linearization of f is admissible near x^* , the quantity $\gamma(\varepsilon)$ (the Lebesgue number of the open covering $\{CM_i\}$ of the compact set P_ε) and the function β can be effectively estimated; in any case

$$\lim_{p \rightarrow 0} \beta(p) = 0.$$

3. The theorem of the preceding section may serve as a basis for constructing approximations to fixed points of a continuous self-mapping of a polyhedron by means of simplicial subdivisions. However, not every fixed point can be approximated in this way; this is not difficult to illustrate by examples for any dimension $n \geq 1$.

Theorem 2. *Let a fixed point x^* of a continuous mapping f of a polyhedron T into itself have nonzero topological index, and let a polyhedral closed neighborhood of this point $U \subset T$ be given, with U containing no other fixed points of the mapping f . Further, let δ be the Lebesgue number of the covering of the boundary of the set U by the family of open sets $(CM_i)_{i \in Z_{n+1}}$. Then, for any subdivision of the polyhedron U , effected by a complex K with diameter less than δ , under any numbering ν constructed for the mapping f , there exists a normal simplex in the complex K .*

The proof uses the theorem on the index from paper (6).

Corollary. *Let the conditions of Theorem 2 be fulfilled, and let K_m be a sequence of complexes with diameters $\delta_m < \delta$, $\delta_m \rightarrow 0$ as $m \rightarrow \infty$, $|K_m| = U$. Then, for any sequence $\{\sigma_m\}$ of representative simplices $\sigma_m \in K_m$, every sequence of points $x_m \in \sigma_m$ converges to the fixed point x^* of the mapping f .*

As a further consequence, we obtain a statement concerning fixed points under the conditions of the Lefschetz-Hopf theorem. In doing so, it is necessary to take into account the presence of different variants of numberings of the points of the polyhedron, associated with its different realizations.

Theorem 3. *Let $f : P \rightarrow P$ be a continuous mapping of a dimensionally homogeneous n -dimensional polyhedron, having only regular fixed points with respect to some complex K , and let the Lefschetz number $\Lambda_f \neq 0$. Then there exists a fixed point $x^* \in P$ to which the representative simplices converge (in the sense of the corollary to Theorem 2).*

The method of subdivisions can be justified for a broader class of problems, above all for the search for boundary or non-isolated fixed points and for point-to-set mappings.

4. We describe one algorithm for finding representative simplices. Let $T = (a_1, a_2, \dots, a_{n+1})$ be an n -dimensional simplex and let K be its triangulation. Construct an abstract complex K_1 , representing a refinement of the complex K . For this we introduce new vertices $\{b_i\}$, $i \in Z_{n+1}$, and for all

For $I \subset Z_{n+1}$, consider the collection of n -dimensional simplices $\tau(I)$, for each of which the vertices are $\{b_i\}$, $i \in CI$, and the vertices of some full-dimensional simplex $\sigma(I) \in K$ lying in the face $\Gamma(I) = (a_i)_{i \in I}$ of the original simplex T . All such simplices $\tau(I)$ and their faces are included in K_1 . It can be shown that K_1 is a closed pseudomanifold. Let the mapping μ , which assigns to the vertices x of the complex K subsets $\mu(x) \subset Z_{n+1}$, be such that, for all vertices x from the subcomplex of the complex K corresponding to the face $\Gamma(I)$, one has $\mu(x) \supset CI$. One way of constructing such a mapping μ is indicated at the end of Section 2. Extend the mapping μ to the set of vertices of K_1 , setting for the new vertices $\mu(b_i) = \{i\}$, and consider some numbering ν of the vertices of the complex K_1 , i.e., a single-valued function such that $\nu(x) \in \mu(x)$. The proposed algorithm is based on

Lemma 2. *If, for a nonempty $I \subset Z_{n+1}$, some simplex $\tau(I) \in K_1$ is normal in the numbering ν , then the corresponding simplex $\sigma(I) \in K$ is representative for μ .*

In the algorithm, as the initial simplex one takes the simplex $\tau(\emptyset) = (b_1, b_2, \dots, b_{n+1})$, which is, obviously, normal. Suppose now that there is some normal simplex $\sigma_0 \in K_1$. Fix $j \in Z_{n+1}$ and construct n -dimensional simplices $\sigma_q \in K_1$, each of which is obtained from the preceding one by deleting some vertex w_{q-1} and including a vertex w'_q distinct from it (uniquely determined), where $\nu(w_0) = j$, and, if $\nu(w'_q) \neq j$, then as w_q one takes the unique vertex of σ_q such that $\nu(w_q) = \nu(w'_q)$, $w_q \neq w'_q$. From the properties of pseudomanifolds and from the fact that a normal simplex contains one, and the others contain 0 or 2, (Cj) -normal faces, it follows that after a finite number of steps we obtain $\nu(w'_q) = j$. The simplex σ_q corresponding to it and distinct from σ_0 is normal, and Lemma 2 is applicable to it.

The algorithm described has common features with the method proposed and successfully tested (for $n \sim 10$) in [7] for approximating fixed points under the conditions of Brouwer's theorem. It can be shown that the family of primitive sets from [7] forms a simplicial complex K such that the polyhedron $|K|$ is close to the original simplex T , and the algorithm in [7] is essentially one of the

economical methods for finding a representative simplex in K . Therefore one cannot agree with the opposition of the algorithm from [7] to a method based on Sperner's lemma. Moreover, from Theorem 2 given above one can obtain the sufficient condition, absent in [7], for the applicability of the method discussed there.

In conclusion, we note that the information report [8] mentions a recent talk by G. Kuhn devoted to the approximation of fixed points of a mapping of a simplex into itself by means of simplicial subdivisions.

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