

ON EVERYWHERE DIVERGENT EXTENDED HERMITE- FEJÉR INTERPOLATION PROCESSES

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Abstract

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MATHEMATICS

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(Presented by Academician L. V. Kantorovich on 29 XII 1969)

1°. Introduction.

In 1916 L. Fejér ⁽¹⁾ proved the following important theorem.

Let $f(x)$ be an arbitrary continuous function on the segment $[-1, 1]$, and let $H_n(f, x)$ be the polynomial of degree $(2n - 1)$, uniquely determined by the conditions

$$H_n(f, x_k^{(n)}) = f(x_k^{(n)}), \quad H'_n(f, x_k^{(n)}) = 0, \quad k = 1, 2, \dots, n,$$
$$x_k^{(n)} = \cos \frac{2k-1}{2n} \pi. \quad (1)$$

Then the relation

$$H_n(f, x) \rightarrow f(x), \quad n \rightarrow \infty$$

holds uniformly on $[-1, 1]$.

Let us now consider the so-called extended Hermite-Fejér interpolation polynomial $F_n(f, x)$ of degree $(2n + 3)$, which is uniquely determined by the equalities

$$F_n(f, 1) = f(1), \quad F_n(f, -1) = f(-1), \quad F'_n(f, 1) = 0, \quad (2)$$

$$F'_n(f, -1) = 0, \quad F_n(f, x_k^{(n)}) = f(x_k^{(n)}), \quad F'_n(f, x_k^{(n)}) = 0, \quad k = 1, 2, \dots, n,$$

where $\{x_k^{(n)}\}_{k=1}^n$ are the Chebyshev nodes (1). In (2, 3) it was proved that the process constructed even for such a simple function as $|x|$ diverges at $x = 0$. Since the node matrix of the process $\{F_n\}_{n=1}^\infty$ is obtained by extending the node matrix of the process $\{H_n\}_{n=1}^\infty$ by adding the points ± 1 as nodes, this result, in view of the aforementioned theorem of L. Fejér, is unexpected. We now weaken conditions (2); namely, we consider the polynomial $A_n(f)$ of degree $(2n + 2)$, which is uniquely determined by the equalities

$$\begin{aligned} A_n(f, 1) &= f(1), & A_n(f, -1) &= f(-1), & A'_n(f, 1) &= 0, \\ A_n(f, x_k^{(n)}) &= f(x_k^{(n)}), \\ A'_n(f, x_k^{(n)}) &= 0, & k &= 1, 2, \dots, n, \end{aligned} \quad (3)$$

where $x_k^{(n)}$ are, as before, the nodes (1). The polynomials A_n are in a certain sense closer to the polynomials H_n than are the polynomials F_n . Therefore the question of whether the process $\{A_n(f)\}_{n=1}^\infty$ will converge uniformly for every function continuous on $[-1, 1]$ is of definite interest. Recently R. B. Saxena (4), with the aid of results from (3), proved simply that the process $\{A_n(f)\}_{n=1}^\infty$ diverges at $x = 0$ if $f(x) = |x|$. Thus here the same situation occurs as in the case of the process $\{F_n(f)\}_{n=1}^\infty$.

In connection with the above, the following questions arise:

1. Does there exist a function continuous on $[-1, 1]$ for which the process $\{A_n(f, x)\}_{n=1}^\infty$ diverges at all points of $(-1, 1)$?
2. Does there exist a function continuous on $[-1, 1]$ for which the process $\{F_n(f, x)\}_{n=1}^\infty$ diverges at all points of $(-1, 1)$?

In this note only the first question is considered. The second question is resolved in an analogous way.

2°. Formulation and proof of the theorem.

Theorem 1. *The interpolation process $\{A_n(f, x)\}_{n=1}^\infty$, constructed at the nodes (1) for $f(x) = 1 - x^2$, diverges at all points of the interval $(-1, 1)$.*

Proof. We need the following

Lemma. *For any $\theta \in [0, \pi/2]$ one can find a sequence of natural numbers $n_1 < n_2 < \dots$, $n_k \rightarrow \infty$, $k \rightarrow \infty$, such that the equality*

$$\lim_{k \rightarrow \infty} \sin^2 n_k \theta = 0$$

holds.

The polynomial $A_n(f, x)$, uniquely determined by the conditions (2), has the form

$$A_n(f, x) = \left[\left(\frac{1-x}{2} \right)^2 f(-1) + \frac{1+x}{2} \left\{ 1 + \frac{4n^2+1}{2}(1-x) \right\} f(1) \right] T_n^2(x) + \sum_{\nu=1}^n f(x_\nu^{(n)}) \frac{(1-x^2)(1-x)}{(1-x_\nu^2)(1-x_\nu)} \frac{(1-x_\nu^2) + (1+2x_\nu)(x-x_\nu)}{n^2(x-x_\nu)^2} T_n^2(x),$$

$$T_n(x) = \cos n \arccos x. \quad (4)$$

Therefore, for $f(x) = 1 - x^2$ we have:

$$A_n(f, x) = \frac{(1-x^2)(1-x)T_n^2(x)}{n^2} \left[\sum_{\nu=1}^n \frac{1+x_\nu}{(x-x_\nu)^2} + \sum_{\nu=1}^n \frac{1+2x_\nu}{(1-x_\nu)(x-x_\nu)} \right]. \quad (5)$$

Let us note that

$$\frac{1+2x_\nu}{(1-x_\nu)(x-x_\nu)} = \frac{1+2x}{(1-x)(x-x_\nu)} + \frac{3}{(x-1)(1-x_\nu)}.$$

Consequently, (5) can be written in the form

$$A_n(f, x) = \frac{(1-x^2)(1-x)T_n^2(x)}{n^2} \left[\sum_{\nu=1}^n \frac{1}{(x-x_\nu)^2} + \sum_{\nu=1}^n \frac{x}{(x-x_\nu)^2} - \sum_{\nu=1}^n \frac{1}{x-x_\nu} + \frac{1+2x}{1-x} \sum_{\nu=1}^n \frac{1}{x-x_\nu} + \frac{3}{x-1} \sum_{\nu=1}^n \frac{1}{1-x_\nu} \right]. \quad (6)$$

It is known that

$$\frac{T_n'(x)}{T_n(x)} = \sum_{\nu=1}^n \frac{1}{x-x_\nu}, \quad \frac{(1-x^2)T_n^2(x)}{n^2} \sum_{\nu=1}^n \frac{1}{(x-x_\nu)^2} = 1 - \frac{\sin 2n\theta \cos \theta}{2n \sin \theta}, \quad x = \cos \theta,$$

$$\sum_{\nu=1}^n \frac{1}{1-x_\nu} = n^2. \quad (7)$$

Therefore from (6) we derive

$$A_n(f, x) = (1-x^2) + \frac{\sin 2n\theta \sin 2\theta}{2n} - 3 \sin^2 \theta \cos^2 n\theta, \quad x = \cos \theta. \quad (8)$$

Consider first the case when $0 \leq x < 1$. Suppose that, for some x in $[0, 1)$, convergence of the process $\{A_n(f, x)\}_{n=1}^\infty$ takes place. Then, by (8), we have

$$\sin 2n\theta \sin 2\theta / 2n - 3 \sin^2 \theta \cos^2 n\theta \rightarrow 0, \quad n \rightarrow \infty. \quad (9)$$

This equality is equivalent to the equality

$$\lim_{n \rightarrow \infty} \cos^2 n\theta = 0, \quad \theta \in (0, \pi/2],$$

which contradicts the lemma. Consequently, in $[0, 1)$ the process diverges.

Consider now the interval $(-1, 0)$. If at some point $\tilde{x} \in (-1, 0)$ the process $\{A_n\}_{n=1}^\infty$ converged, then, according to the preceding, $\lim_{n \rightarrow \infty} \cos^2 n\bar{\theta} = 0$ (9), where $\tilde{x} = \cos \bar{\theta}$. Put $\theta = \pi - \bar{\theta}$, $\pi/2 < \bar{\theta} < \pi$. Thus $0 < \theta < \pi/2$. Since

$$\cos^2 n\bar{\theta} = \cos^2 n\theta,$$

it follows from (9) that $\lim_{n \rightarrow \infty} \cos n\theta = 0$. This again contradicts the lemma. Thus the process $\{A_n(f)\}_{n=1}^\infty$ also diverges on the interval $(-1, 0)$. The theorem is proved.

Along with the polynomial $A_n(f, x)$, consider the polynomial $B_n(f, x)$, uniquely determined by the conditions

$$B_n(f, 1) = f(1), \quad B_n(f, -1) = f(-1), \quad B'_n(f, -1) = 0,$$

$$B_n(f, x_k^{(n)}) = f(x_k^{(n)}), \quad B'_n(f, x_k^{(n)}) = 0, \quad k = 1, 2, \dots, n.$$

By an almost verbatim repetition of the arguments in the proof of Theorem 1, we obtain Theorem 2.

Theorem 2. *The interpolation process $\{B_n(f, x)\}_{n=1}^\infty$, constructed at the Chebyshev nodes (1) for $f(x) = 1 - x^2$, diverges at all points of the interval $(-1, 1)$.*

Corollary. *The interpolation process $\{A_n(f)\}_{n=1}^\infty$, constructed at the Chebyshev nodes (1) for $f(x) = x^2$, diverges at all points of the interval $(-1, 1)$.*

Indeed, for $f \equiv 1$, $A_n(f) \equiv 1$, hence

$$A_n(1 - z^2, x) = 1 - A_n(z^2, x). \quad (10)$$

If, for $x \in (-1, 1)$, $A_n(z^2, x) \rightarrow x^2$, $n \rightarrow \infty$, then, according to (10), $A_n(1 - z^2, x) \rightarrow 1 - x^2$, $n \rightarrow \infty$, which contradicts Theorem 1.

Theorem 3. *The interpolation process $\{A_n(f)\}_{n=1}^\infty$, constructed at the Chebyshev nodes (1) for $f(x) = x$, diverges at all points of the interval $(-1, 1)$.*

Proof. Since $A_n(1, x) \equiv 1$, according to (4),

$$1 - A_n(z, x) = T_n^2(x) \left[\frac{(x-1)^2}{2} + \frac{(1-x)(1-x^2)}{n^2} \sum_{\nu=1}^n \frac{(1-x_\nu^2) + (1+2x_\nu)(x-x_\nu)}{(1-x_\nu^2)(x-x_\nu)^2} \right].$$

Hence, after simple transformations, we obtain

$$1 - A_n(z, x) = T_n^2(x) \left[\frac{(x-1)^2}{2} + \frac{(1-x)(1-x^2)}{n^2} \left(\frac{3}{2(x-1)} \sum_{\nu=1}^n \frac{1}{1-x_\nu} - \sum_{\nu=1}^n \frac{1}{2(x+1)(1+x_\nu)} + \frac{2x+1}{1-x^2} \sum_{\nu=1}^n \frac{1}{x-x_\nu} + \sum_{\nu=1}^n \frac{1}{(x-x_\nu)^2} \right) \right]. \quad (11)$$

Expression (11), after application of identity (7) and the identity

$$\sum_{\nu=1}^n 1/(1+x_\nu) = n^2$$

takes the form

$$1 - A_n(z, x) = (1-x) - \frac{(1-x) \sin 2n\theta \cos \theta}{2n \sin \theta} + \frac{3(x^2+1) \cos^2 n\theta}{2} + \frac{(2x+1)(1-x) \cos n\theta \sin n\theta}{n \sin \theta}, \quad x = \cos \theta. \quad (12)$$

If $A_n(z, x) \rightarrow x$, $n \rightarrow \infty$, then from (12) it would follow that $\lim_{n \rightarrow \infty} \cos^2 n\theta = 0$, and this contradicts the lemma.

Remark. In connection with Theorems 1 and 3 it is curious that if the process $\{A_n(f)\}_{n=1}^\infty$ is constructed for $f(x) = (1-x^2)(1-x)$, then it converges uniformly. Moreover,

$$|A_n(f) - f| \leq 12/n, \quad |x| \leq 1, \quad f(x) = (1-x^2)(1-x). \quad (13)$$

Indeed, in the present case

$$A_n(f, x) = \frac{(1-x)(1-x^2)T_n^2(x)}{n^2} \sum_{\nu=1}^n \left(\frac{1-x^2}{(x-x_\nu)^2} - \frac{4x+1}{x-x_\nu} - 3 \right).$$

Hence, with the aid of the identities (7), we obtain

$$A_n(f, x) = (1-x)(1-x^2) - \frac{(1-x) \sin 2n\theta \sin 2\theta}{4n} - \frac{(4x+1)(1-x) \sin 2n\theta \sin \theta}{2n} - \frac{3(1-x) \cos^2 n\theta \sin^2 \theta}{n}.$$

Thus, (13) holds.

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CITED LITERATURE

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