

MODULES AND MEASURABILITY OF SURFACES

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Abstract

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MATHEMATICS

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MODULES AND MEASURABILITY OF SURFACES

(Presented by Academician M. A. Lavrent'ev 4 I 1970)

In this note we study the modules of various families of curves and k -dimensional surfaces ($2 \leq k \leq n-1$) passing through a point of Euclidean n -dimensional space R^n , by means of which the measurability of k -dimensional surfaces in the sense of Hausdorff is investigated.

Points of the space R^n are denoted by P or $x = (x_1, x_2, \dots, x_n)$, and for a given set $E \subset R^n$ by $\Lambda^k(E)$, $1 \leq k \leq n$, the k -dimensional Hausdorff measure.

A surface s (k -dimensional) will be called Lipschitz at a point $P \in s$ if there exists a neighborhood of it on s that is the image under a Lipschitz mapping of a certain domain of the space R^k .

Let S^k be some family of k -dimensional surfaces of the space R^n , and let $M(S^k)$ be its module ⁽¹⁾. The family S^k is called exceptional if $M(S^k) = 0$. It is said that a certain property holds for almost all surfaces of the family S^k if the subfamily S^k on which it does not hold is exceptional.

Let (r, γ) be polar coordinates in the x_1x_2 -plane of the space R^3 , and let $\gamma = \gamma(r)$, $r_1 \leq r \leq r_2$, be a smooth curve lying in the half-plane $x_2 > 0$, except possibly for its end $\gamma_1 = \gamma(r_1)$, and let S_γ^0 be the surface formed by rotating $\gamma(r)$ about the x_1 -axis. Then the following basic result holds.

Lemma. If S_γ^2 is the family of all possible surfaces s_γ in the space R^3 obtained from s_γ^0 by rotation about the point $P = 0$, then

$$M(S_\gamma^2) = 2 \left[2\pi \int_{r_1}^{r_2} \frac{\sin^3 \gamma(r)}{r} \{1 + r^2 [\gamma'(r)]^2\}^{3/2} dr \right]^{-1/2}, \quad (1)$$

where $\gamma'(r) = d\gamma/dr$.

We give the scheme of the proof. First it is established that, in finding the module of the family S_γ^2 , it suffices to restrict oneself to metrics of the form $\rho = \rho(r)$, $r = |x|$. Then the metric is introduced

$$\rho_0 = \rho_0(r) = \begin{cases} \left(\int_{r_1}^{r_2} \rho_1^3 r^2 dr \right)^{-1/2} \rho_1, & r \in [r_1, r_2], \\ 0, & r \notin [r_1, r_2], \end{cases}$$

where $\rho_1 = \rho_1(r) = \sigma'(r)/r^2$, $\sigma'(r) = d\sigma/dr$;

$$\sigma = \sigma(r) = 2\pi \int_{r_1}^r r \sin \gamma(r) \times \{1 + r^2[\gamma'(r)]^2\}^{1/2} dr$$

is the part of the area of the surface s_γ^0 corresponding to the interval $[r_1, r]$.

The metric ρ_0 is admissible for the family S_γ^2 , since

$$\int_{s_\gamma} \rho_0^2 d\Lambda^2 = \int_{s_\gamma^0} \rho_0^2 d\Lambda^2 = \int_0^{\sigma(r_2)} \rho_0^2 d\sigma = \int_{r_1}^{r_2} \rho_0^2 \sigma'(r) dr = 1.$$

Moreover,

$$\int_{R^3} \rho_0^3 dV = \int_0^{2\pi} dr \int_0^\pi \sin \theta d\theta \int_{r_1}^{r_2} \rho_0^3 r^2 dr = 4\pi \left(\int_{r_1}^{r_2} \rho_1^3 r^2 dr \right)^{-1/2},$$

where (r, θ, φ) are spherical coordinates in the space R^3 .

On the other hand, for an arbitrary metric $\rho = \rho(r)$ admissible for the family S_γ^2 , by Hölder's inequality we have

$$1 \leq \int_{s_\gamma} \rho^2 d\Lambda^2 = \int_{r_1}^{r_2} \rho^2 \sigma'(r) dr \leq \left(\int_{r_1}^{r_2} \rho^3 r^2 dr \right)^{2/3} \left(\int_{r_1}^{r_2} \rho_1^3 r^2 dr \right)^{1/3}.$$

and, consequently,

$$\int_{R^3} \rho^3 dV = 4\pi \int_{r_1}^{r_2} \rho^3 r^2 dr \geq 4\pi \left(\int_{r_1}^{r_2} \rho_1^3 r^2 dr \right)^{-1/2}.$$

Thus the metric ρ_0 is extremal for the family S_γ^2 , i.e.,

$$M(S_\gamma^2) = 4\pi \left(\int_{r_1}^{r_2} \rho_1^3 r^2 dr \right)^{-1/2},$$

which, after substituting the values $\rho_1 = \sigma'(r)/r^2$ and $\sigma'(r)$, gives (1).

We shall say that the surface s_γ^0 with generator $\gamma = \gamma(r)$, $0 \leq r \leq r_2$, has a cusp at the point $P = 0$ if $\gamma(0) = 0$ or $\gamma(0) = \pi$.

Let us note some consequences of the lemma.

Corollary 1. If in (1) $\gamma(r) = \gamma_0 = \text{const}$, then

$$M(S_\gamma^2) = 2(2\pi \sin^3 \gamma_0 \ln r_2/r_1)^{-1/2}.$$

Corollary 2. The family S_γ^0 , $0 \leq r \leq r_2$, of surfaces having a cusp at the point $P = 0$, is nonexceptional.

Corollary 3. The family S_γ^2 , $0 \leq r \leq r_2$, of surfaces not having a cusp at the point $P = 0$, is nonexceptional.

The following theorem, strengthening theorem 5 (2), shows that for the exceptionality of a family of k -dimensional surfaces passing through a point, it suffices that its surfaces satisfy certain restrictions only in a neighborhood of the point through which they pass.

Theorem 1. The family of all possible k -dimensional surfaces ($2 \leq k \leq n - 1$) passing through a point P of the space R^n and being Lipschitz at P , is exceptional.

On the other hand, there exist sufficiently broad families S^k of k -dimensional surfaces passing through a point, for which $0 < M(S^k) < \infty$. For example, let S_γ^2 be the family of surfaces defined above with generators $\gamma(r) = \arcsin r$, $0 \leq r \leq r_2 < 1$. Then, on the basis of (1),

$$M(S_\gamma^2) = 2\{2\pi[r_2/(1 - r_2^2)^{1/2} - \gamma'(r_2)]\}^{-1/2}.$$

Moreover, by letting r_2 tend here to zero, one can obtain a family with an arbitrarily large modulus.

In connection with the noted effect, the following is true.

Theorem 2. The modulus of the family of all possible k -dimensional surfaces ($2 \leq k \leq n - 1$) passing through a point of the space R^n , is equal to infinity.

For finding the moduli of a family of k -dimensional surfaces, the answer to the question whether surfaces nonmeasurable in the sense of Hausdorff affect the modulus is of essential importance.

The answer to this question is given by

Theorem 3. *Almost all bounded, but not almost all k -dimensional surfaces ($2 \leq k \leq n - 1$) in the space R^n are Hausdorff measurable.*

Let us note that a nonexceptional family of Hausdorff-nonmeasurable surfaces can be constructed with the aid of the inversion transformation $y = 1/x$ from the family S_γ^n of Corollary 2.

For families of curves ($k = 1$) simpler results hold.

Theorem 4. *The family of all curves passing through a point of the space R^n is exceptional.*

The theorem is proved by introducing the metric

$$\rho = \rho(r) = \begin{cases} (r \ln 2/r)^{-1}, & r < 1, \\ 0, & r \geq 1, \end{cases}$$

which is summable in R^n with the n -th power and

$$\int_\gamma \rho d\Lambda = \infty$$

for every curve γ of the family under consideration. The latter, by Theorem 2⁽²⁾, means that the family is exceptional.

Corollary (J. Väisälä⁽³⁾). *Almost all curves in the space R^n are rectifiable.*

Thus, if in finding the moduli of families of k -dimensional surfaces one may neglect only bounded non-Hausdorff-measurable surfaces, then nonrectifiable curves have no influence at all on the modulus.

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REFERENCES

- ¹ B. V. Shabat, DAN, 130, No. 6 (1960).
- ² B. Fuglede, Acta Math., 98, No. 3-4 (1957).
- ³ J. Väisälä, Ann. Acad. Sci. Fenn., Ser. A, 298 (1961).

Note: Figure translations are in progress. See original paper for figures.

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