

ESTIMATES IN THE NORMS $\|(H_{(s)})\|$ AND HYPOELLIPTICITY

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Abstract

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MATHEMATICS

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ESTIMATES IN THE NORMS $H_{(s)}$ AND HYPOELLIPTICITY

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Let P be a linear differential operator with infinitely differentiable coefficients, defined in a domain $\Omega \subseteq \mathbf{R}^n$ (or on a manifold of class C^∞). The operator P is called hypoelliptic if, for every distribution u in Ω , one has

$$\text{sing supp } u = \text{sing supp } Pu.$$

In this paper we give sufficient conditions for hypoellipticity that are more general than those in [1]. With their help it is possible to prove the hypoellipticity of certain second-order differential operators with real coefficients and a semidefinite principal part which do not satisfy the conditions obtained in [4, 6].

For convenience of formulation, let us introduce the conditions A, B, and C written below. As usual, $[P, l]$ denotes the commutator of the operators P and l .

A. For every point $x_0 \in \Omega$ there exist a neighborhood $U(x_0)$ and real numbers s_0 and a such that, whatever the real number N , with some constant C_N independent of u , the inequality

$$\|u\|_{(s_0+a)}^2 \leq C_N \{\|Pu\|_{(s_0)}^2 + \|u\|_{(-N)}^2\}, \quad u \in C_0^\infty(U(x_0)). \quad (1)$$

holds.

Since in inequality (1), for all x belonging to some compact set K , one may take one and the same a , we shall assume that a does not depend on x .

B. For every point $x_0 \in \Omega$ there exist a neighborhood $U(x_0)$ and a real number s_1 such that, whatever the linear differential operator l of order zero or one with infinitely differentiable coefficients in Ω , and whatever the real $s > s_1$, $\varepsilon > 0$, and N , with some constant $C_{\varepsilon, l, N, s}$ independent of u , the inequality

$$\|[P, l]u\|_{(s)}^2 \leq \varepsilon \|Pu\|_{(s+\text{ord } l)}^2 + C_{\varepsilon, l, N, s} \|u\|_{(-N)}^2, \quad u \in C_0^\infty(U(x_0)). \quad (2)$$

holds.

C. For every point $x_0 \in \Omega$ and every neighborhood $U(x_0)$ there exist a real number s_2 and a sequence of functions $\{\psi_j(x)\}$ ($\psi_j(x) \in C_0^\infty(U(x_0))$, $\psi_1(x_0) \neq 0$, ψ_{j+1} is equal to one in a neighborhood of the support of ψ_j) such that, for arbitrary real $s > s_2$ and N , with some $\varkappa > 0$ and constants $C_{j,N,s}$ independent of u , the inequalities

$$\| [P, \psi_j] u \|_{(s)}^2 \leq C_{j,N,s} \{ \| Pu \|_{(s-\varkappa)}^2 + \| u \|_{(-N)}^2 \}, \quad u \in C_0^\infty(U(x_0)). \quad (3)$$

hold.

Theorem 1. *If the operator P satisfies conditions A and B in Ω , then from $u \in \mathcal{E}'(\Omega)$, $Pu \in H_{(s)}(\Omega)$, it follows that $u \in H_{(s+a)}(\Omega)$.*

Consider the operator P , equal to

$$\sum_{j=1}^r X_j^2 + X_0 + c,$$

where all X_j are homogeneous first-order differential operators with real infinitely differentiable coefficients in Ω .

Definition. We shall say that the operator P satisfies Hörmander's condition at the point $x \in \Omega$ if at this point the Lie algebra generated by the operators $X_0, \dots, X_r$ has rank n . In [4] it is proved that,

that if P satisfies this condition at every point of Ω , then it is hypoelliptic.

The following lemma gives an example of operators satisfying the conditions of Theorem 1.

Lemma. Let M be a smooth submanifold in Ω of dimension $\leq n-1$. Suppose that at all points $x \in \Omega \setminus M$ the operator

$$P = \sum_{j=1}^r X_j^2 + X_0 + c$$

satisfies the Hör condition, while at those points $x \in M$ where this condition is violated, at least one of the operators X_j ($j = 1, \dots, r$) is transversal to M . Then the operator P satisfies condition A with $a = 0$ and condition B.

If Ω is a domain in $\mathbb{R}^n$, then condition B is equivalent to condition B', which is more convenient for use.

B'. For any point $x_0 \in \Omega$ there exists a neighborhood $U(x_0)$ such that for any multi-indices α and β , $|\alpha| + |\beta| \neq 0$, and certain real s_β^α , whatever $\varepsilon > 0$ and N may be, the inequalities

$$\| P_{(\beta)}^{(\alpha)} u \|_{(s_\beta^\alpha)}^2 \leq \varepsilon \| Pu \|_{(s_\beta^\alpha + |\beta|)}^2 + C \| u \|_{(-N)}^2, \quad u \in C_0^\infty(U(x_0)). \quad (4)$$

hold with certain constants independent of u , but possibly depending on $x_0, \alpha, \beta, \varepsilon$, and N .

Here $P_{(\beta)}^{(\alpha)}$ is the differential operator corresponding to the symbol $D_\xi^\alpha D_x^\beta p(x, \xi)$, where $p(x, \xi)$ is the symbol of the operator P .

Remark. It can be shown that from the validity of B' it follows that the inequalities (4) hold with arbitrary s in place of s_β^α in the exponents of the norms. Conversely, if the inequalities (4) hold only for $|\alpha| + |\beta| = 1$, but for all s , then condition B' is fulfilled. From the validity of condition B' there follows the validity of condition B'' .

B'' . For any compact $K \subset \Omega$, any multi-indices β , $|\beta| \neq 0$, and certain real s_β , whatever $\varepsilon > 0$ and N may be, the inequalities

$$\|P_{(\beta)} u\|_{(s_\beta)}^2 \leq \varepsilon \|Pu\|_{(s_\beta + |\beta|)}^2 + C \|u\|_{(-N)}^2, \quad u \in C_0^\infty(K). \quad (5)$$

hold with certain constants independent of u .

Theorem 1'. Let the operator P satisfy conditions A and B'' in some domain $\Omega \subseteq \mathbb{R}^n$. Then from $u \in D'(\Omega)$, $Pu \in H_{(s)}^{\text{loc}}(\Omega)$, and $\text{sing supp } u$ being compact in Ω , it follows that $u \in H_{(s+a)}^{\text{loc}}(\Omega)$.

Let us note that conditions A and B'' are satisfied for all differential operators with constant coefficients. Therefore Theorem 1' may be regarded as a certain generalization of a theorem of M. S. Agranovich (⁷).

The following theorem is the main result of the paper.

Theorem 2. If the operator P satisfies conditions A, B, and C in Ω , then from $u \in D'(\Omega)$, $Pu \in H_{(s)}^{\text{loc}}(\Omega')$, where Ω' is an arbitrary open subset in Ω , it follows that $u \in H_{(s+a)}^{\text{loc}}(\Omega')$. Consequently, the operator P is hypoelliptic.

In the proof of Theorems 1 and 2, mollifiers studied in (^{2, 3}) are used.

Theorem 3. If the operator

$$P = \sum_{j=1}^r X_j^2 + X_0 + c$$

at every point of Ω , except for a certain set Q of isolated points, satisfies the Hör condition, and at every point $x \in Q$ at least one of the operators X_j ($j = 0, 1, \dots, r$) is nonzero, then from $u \in D'(\Omega)$, $Pu \in H_{(s)}^{\text{loc}}(\Omega')$, it follows that $u \in H_{(s)}^{\text{loc}}(\Omega')$. Consequently, the operator P is hypoelliptic.

Theorem 4. Consider the differential operator $P = -\partial^2/\partial x^2 - \varphi^2(x)\partial^2/\partial y^2$. Suppose that $\varphi(x)$ belongs to $C^\infty(\mathbf{R}^1)$, is equal to zero at $x = 0$ together with

all its derivatives, and is different from zero for $x \neq 0$. Then from $u \in D'(\Omega)$, $Pu \in H_{(s)}^{\text{loc}}(\Omega')$, it follows that $u \in H_{(s)}^{\text{loc}}(\Omega')$. Consequently, the operator P is hypoelliptic.

Theorems 3 and 4 are proved by using Theorem 2. The greatest difficulty is the verification that the operator $-\partial^2/\partial x^2 - \varphi^2(x)\partial^2/\partial y^2$ satisfies condition C. By Fourier transformation with respect to y , this problem is reduced to proving the estimate

$$\left\| |\eta|^{\varkappa} \frac{d}{dx} \varphi^2 u \right\|_{(0)}^2 + \left\| |\eta|^{1+\varkappa} \varphi^2 u \right\|_{(0)}^2 \leq C \left\{ \|Q_\eta u\|_{(0)}^2 + \|u\|_{(0)}^2 \right\}, \quad u \in C_0^\infty(K). \quad (6)$$

Here K is some compact set in $\mathbf{R}^1$, $Q_\eta = -d^2/dx^2 + \varphi^2(x)\eta^2$, the constant C does not depend on η , and $\varkappa$ is some real number greater than zero. First, estimate (6) is proved when the support of u belongs to the set $\{x : |\varphi(x)| > b/\sqrt{|\eta|}\}$, where b is a certain constant depending on the function $\varphi(x)$. In doing so, the method of paper (5) is used. Then estimate (6) is proved by integration by parts when the support of u belongs to the set $\{x : |\varphi(x)| < 2b/\sqrt{|\eta|}\}$. The proof is completed by using a certain special partition of unity.

Remark. Theorems 1 and 2 are also valid for pseudodifferential operators with compact supports.

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