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Abstract

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MATHEMATICS

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ON SYMBOLS OF ONE-DIMENSIONAL SINGULAR INTEGRAL OPERATORS ON AN OPEN CONTOUR

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1. A one-dimensional singular integral operator is an operator A defined by the equality

$$(A\varphi)(t) = c(t)\varphi(t) + \frac{d(t)}{\pi i} \int_{\Gamma} \frac{\varphi(\tau)}{\tau - t} d\tau \quad (t \in \Gamma), \quad (1)$$

where Γ is a contour in the complex plane; $c(t)$ and $d(t)$ ($t \in \Gamma$) are prescribed functions (or matrix-functions), which are called the coefficients of the operator A .

In what follows it will be assumed that the contour Γ consists of a finite number of closed and open simple contours of Lyapunov type. Singular integral operators will be considered in the Hilbert space $L_2(\Gamma)$, and in the case where $c(t)$ and $d(t)$ are matrix-functions of order n , in the space $L_2^n(\Gamma)$ of vector-functions $\varphi = \{\varphi_j\}_{j=1}^n$, where $\varphi_j \in L_2(\Gamma)$.

Denote by $\mathfrak{A}_n$ the smallest subalgebra of the Banach algebra $\mathfrak{R}_n$ of all linear bounded operators acting in the space $L_2^n(\Gamma)$, containing all operators of the form (1) with piecewise-continuous coefficients.

In the present note it is proved that the factor algebra $\mathfrak{A}_n/\mathfrak{S}_\infty$, where $\mathfrak{S}_\infty$ is the ideal of all completely continuous operators in $\mathfrak{R}_n$, is isomorphic and isometric to a certain algebra of matrix-functions of order $2n$, defined on the cylinder $\mathfrak{M} = \{(t, \mu) : t \in \Gamma, 0 \leq \mu \leq 1\}$. This isomorphism assigns to each operator in $\mathfrak{A}_n$ its symbol. It is also proved that an operator $A \in \mathfrak{A}_n$ is a Fredholm operator if and only if its symbol is a nonsingular matrix at every point of the cylinder. A formula is established for computing the index of Fredholm operators belonging to the algebra $\mathfrak{A}_n$. For the case when the contour Γ consists only of closed lines, these results were obtained by the authors in paper ⁽¹⁾.

2. We shall agree on the following notation: $\Lambda = \Lambda(\Gamma)$ is the set of all functions piecewise-continuous on Γ which are continuous at the ends of the open arcs of

the contour Γ and left-continuous on the whole contour Γ ; Λ_n is the set of all matrix-functions of order n with elements from Λ ; S is the operator of singular integration along Γ :

$$(S\varphi)(t) = \frac{1}{\pi i} \int_{\Gamma} \frac{\varphi(\tau)}{\tau - t} d\tau \quad (t \in \Gamma),$$

acting in the space $L_2(\Gamma)$, and S_n is the operator of singular integration in $L_2^n(\Gamma)$, i.e. $S_n\{\varphi_j\}_{j=1}^n = \{S\varphi_j\}_{j=1}^n$. The operator $A = c(t)I + d(t)S$, where $c(t), d(t) \in \Lambda_n$, will be conveniently written in the form

$$A = F(t)P + G(t)Q, \quad (2)$$

where $F(t) = c(t) + d(t)$, $G(t) = c(t) - d(t)$, $P = (I + S_n)/2$, and $Q = (I - S_n)/2$.

Suppose the contour Γ , along with closed lines, contains N open arcs, whose beginnings and ends we denote respectively by α_k and β_k ($k = 1, \dots, N$).

By the **symbol** of the operator A , defined by equality (2), we shall mean the matrix-function $\mathcal{A}(t, \mu)$ of order $2n$, defined on the cylinder $\mathfrak{M}$ by the equality

$$\mathcal{A}(t, \mu) = \begin{cases} \begin{pmatrix} F(\alpha_k)\mu + G(\alpha_k)(1 - \mu) & 0 \\ 0 & F(\alpha_k)\mu + G(\alpha_k)(1 - \mu) \end{pmatrix}, & \text{for } t = \alpha_k, \\ \begin{pmatrix} F(t+0)\mu + F(t)(1 - \mu) & \sqrt{\mu(1 - \mu)}(G(t+0) - G(t)) \\ \sqrt{\mu(1 - \mu)}(F(t+0) - F(t)) & G(t+0)(1 - \mu) + G(t)\mu \end{pmatrix}, & \text{for } t \in \Gamma, \quad t = \alpha_k, \beta_k, \\ \begin{pmatrix} F(\beta_k)(1 - \mu) + G(\beta_k)\mu & 0 \\ 0 & F(\beta_k)(1 - \mu) + G(\beta_k)\mu \end{pmatrix}, & \text{for } t = \beta_k. \end{cases}$$

Theorem 1. In order that the operator $A = F(t)P + G(t)Q$, where $F(t), G(t) \in \Lambda_n$, be a Φ -operator in $L_2^n(\Gamma)$, it is necessary and sufficient that its symbol be nowhere degenerate, i.e.

$$\det \mathcal{A}(t, \mu) \neq 0 \quad (t \in \Gamma, 0 \leq \mu \leq 1). \quad (3)$$

Proof. Complete the contour Γ to a contour $\tilde{\Gamma}$, consisting of a finite number of closed simple oriented Lyapunov-type curves. Let $\tilde{F}(t)$ and $\tilde{G}(t)$ ($t \in \tilde{\Gamma}$) be certain matrix-functions coinciding respectively with $F(t)$ and $G(t)$ on Γ , continuous on the closed set $\overline{\tilde{\Gamma} \setminus \Gamma}$, and nondegenerate at every point of the open set $\tilde{\Gamma} \setminus \Gamma$. By $\tilde{S}_n$ denote the matrix operator of singular integration along $\tilde{\Gamma}$, and by $\tilde{P}$ and $\tilde{Q}$ the operators $(I + \tilde{S}_n)/2$, $(I - \tilde{S}_n)/2$, respectively.

Consider the operator

$$\tilde{A} = B_n(\tilde{F}\tilde{P} + \tilde{G}\tilde{Q})B_n + C_n(\tilde{G}\tilde{P} + \tilde{F}\tilde{Q})C_n,$$

where $B_n = \|\delta_{jk}B\|_{j,k=1}^n$, $C_n = I - B_n$, and B is the operator of multiplication in $L_2(\tilde{\Gamma})$ by the characteristic function of the set Γ . The operator $\tilde{A}$ is a sum of products of singular integral operators in $L_2^n(\Gamma)$ with piecewise-continuous matrix coefficients. In [1] the symbol $\tilde{\mathcal{A}}(t, \mu)$ of such an operator is defined. From this definition, in particular, it follows that $\det \tilde{\mathcal{A}}(t, \mu) = \det \mathcal{A}(t, \mu)$, if $t \in \Gamma$, and $\det \tilde{\mathcal{A}}(t, \mu) = \det \tilde{G}(t)\tilde{F}(t)$, if $t \in \tilde{\Gamma} \setminus \Gamma$.

Let condition (3) be satisfied; then $\det \tilde{\mathcal{A}}(t, \mu) \neq 0$ ($t \in \Gamma$, $0 \leq \mu \leq 1$), hence [1], the operator $\tilde{A}$ is a Φ -operator in $L_2^n(\tilde{\Gamma})$, and, consequently, the operator A is a Φ -operator in $L_2^n(\Gamma)$.

Conversely, suppose the operator A is a Φ -operator in $L_2^n(\tilde{\Gamma})$; then the operator $A_1 = B_n\tilde{A}B_n + C_n$ is a Φ -operator in $L_2^n(\tilde{\Gamma})$, and, consequently, $\det \mathcal{A}_1(t, \mu) \neq 0$ ($t \in \tilde{\Gamma}$, $0 \leq \mu \leq 1$). It is not difficult to verify that if $\det \mathcal{A}_1(t, \mu) \neq 0$, then the symbol $\mathcal{A}_2(t, \mu)$ of the operator $A_2 = B_n + C_n\tilde{A}C_n$ is also everywhere nondegenerate, and, consequently, the operator A_2 is a Φ -operator in $L_2(\tilde{\Gamma})$. It follows that the operator $\tilde{A}$ is a Φ -operator in $L_2(\Gamma)$. Since $\det \tilde{\mathcal{A}}(t, \mu) = \det \mathcal{A}(t, \mu)$ for points $t \in \Gamma$, we have $\det \mathcal{A}(t, \mu) \neq 0$ ($t \in \Gamma$, $0 \leq \mu \leq 1$). The theorem is proved.

3. Introduce on the cylinder $\mathfrak{M} = \{(t, \mu) : t \in \Gamma, 0 \leq \mu \leq 1\}$ a topology by defining neighborhoods of each point by one of the five equalities:

$$u(\alpha_k, 0) = \{(\alpha_k, \mu) : 0 \leq \mu < \varepsilon\}; \quad u(\beta_k, 1) = \{(\beta_k, \mu) : \varepsilon < \mu \leq 1\},$$

$$u(t_0, 0) = \{(t, \mu) : |t - t_0| < \delta, t < t_0, 0 \leq \mu \leq 1\} \cup \{(t_0, \mu) : 0 \leq \mu < \varepsilon\} \quad (t_0 \neq \alpha_k),$$

$$u(t_0, 1) = \{(t, \mu) : |t - t_0| < \delta, t > t_0, 0 \leq \mu \leq 1\} \cup \{(t_0, \mu) : \varepsilon < \mu \leq 1\} \quad (t_0 \neq \beta_k),$$

$$u(t_0, \mu_0) = \{(t_0, \mu) : \mu - \delta_1 < \mu < \mu_0 + \delta_2\} \quad (\mu_0 \neq 0; 1),$$

where $0 < \delta_1 < \mu_0$, $0 < \delta_2 < 1 - \mu_0$, $0 < \varepsilon < 1$, and $t < t_0$ means that on the oriented contour Γ the point t precedes the point t_0 .

By $\mathcal{P}_n$ we denote the algebra of matrices of functions of order $2n$ of the form

$$H(t, \mu) = (H_{jk}(t, \mu))_{j,k=1}^2,$$

where $H_{jk}(t, \mu)$ are arbitrary matrix-functions of order n satisfying the following conditions: a) the matrix-functions $H_{11}(t, \mu)$, $H_{12}(t, \mu)$, $H_{21}(t, \mu)$ and $H_{22}(t, 1 - \mu)$ are continuous on the cylinder $\mathfrak{M}$ with the topology introduced above; b) the matrices $H_{21}(t, \mu)$ and $H_{12}(t, \mu)$ vanish if μ takes one of the values 0, 1, and t is an arbitrary point of the contour, and also when t takes one of the values α_k, β_k ($k = 1, \dots, N$), while μ is any number of the interval $0 \leq \mu \leq 1$.

The algebra $\mathcal{P}$ becomes a Banach algebra if one introduces in it the norm equal to

$$\|H(t, \mu)\| = \sup_{\substack{t \in \Gamma \\ 0 \leq \mu \leq 1}} s_1(H(t, \mu)),$$

where the number $[s_1(H(t, \mu))]^2$ for each point of the cylinder $\mathfrak{M}$ denotes the largest eigenvalue of the matrix $H(t, \mu)(H(t, \mu))^*$.

Theorem 2. Let A_{jl} ($j = 1, \dots, k$; $l = 1, \dots, m$) be singular integral operators with matrix coefficients from Λ_n , and let $\mathcal{A}_{jl}(t, \mu)$ be their symbols. Then for the operator

$$A = \sum_{j=1}^k \prod_{l=1}^m A_{jl} \quad (4)$$

the equality

$$\inf_{T \in \mathfrak{S}_\infty} \|A + T\| = \|\mathcal{A}(t, \mu)\| \quad (5)$$

holds, where

$$\mathcal{A}(t, \mu) = \sum_{j=1}^k \prod_{l=1}^m \mathcal{A}_{jl}, \quad (6)$$

and the norm on the right-hand side of (5) is the norm in the algebra $\mathcal{P}_n$.

The proof is analogous to the proof of Theorem 2.2 in ⁽¹⁾.

The matrix-function (6) is naturally called the **symbol** of the operator (4). From equality (5) it follows that the symbol of an operator A does not depend on the manner in which the operator A is represented in the form (4).

Let $\mathfrak{A}_n$ be the algebra obtained by closing the set of operators of the form (4) in the algebra $\mathfrak{R}_n$. Equality (5) makes it possible to define the symbol $\mathcal{A}(t, \mu)$ for each operator $A \in \mathfrak{A}_n$ as the limit in the algebra $\mathcal{P}_n$ of a sequence of symbols $\mathcal{A}_r(t, \mu)$ of operators A_r of the form (4), converging uniformly to the operator A .

Theorem 3. The two-sided ideal $\mathfrak{S}_\infty$ of all completely continuous operators acting in $L_2^n(\Gamma)$ is contained in the algebra $\mathfrak{A}_n$, and the quotient algebra $\mathfrak{A}_n/\mathfrak{S}_\infty$ is isomorphic and isometric to the algebra $\mathcal{P}_n$. Under this isomorphism the residue class containing the operator A ($\in \mathfrak{A}_n$) is mapped to the symbol $\mathcal{A}(t, \mu)$ of the operator A . In order that the operator A ($\in \mathfrak{A}_n$) be a Φ -operator in $L_2^n(\Gamma)$, it is necessary and sufficient that the condition $\det \mathcal{A}(t, \mu) \neq 0$ hold for all $t \in \Gamma$ and $0 \leq \mu \leq 1$.

The proof of this theorem is carried out according to the same scheme as the proof of Theorems 4.1 and 4.2 in ⁽¹⁾.

4. Let us establish a formula for computing the index of Φ -operators from the algebra $\mathfrak{A}_n$. Recall that the index of a Φ -operator A is the number $\text{ind } A$, equal to the difference $\dim \ker A - \dim \text{coker } A$.

In this section we additionally assume that the contour Γ can be completed to a closed contour $\hat{\Gamma}$, bounding a connected set $\hat{M}$ of points of the plane and, consequently, consisting of a finite number of closed simple contours of Lyapunov type. We shall also assume that the point $z = 0$ is an interior point of the set $\hat{M}$.

Theorem 4. *Let the operator $A \in \mathfrak{A}_n$, and let the matrix-function*

$$\mathcal{A}(t, \mu) = \|H_{jk}(t, \mu)\|_{j,k=1}^2$$

*be its symbol. If $\det \mathcal{A}(t, \mu) \neq 0$ ($t \in \Gamma$, *

$0 \leq \mu \leq 1$), then the function

$$f_A(t, \mu) = \begin{cases} \det H_{22}(\alpha_k, \mu) H_{22}^{-1}(\alpha_k, 0), & \text{for } t = \alpha_k, \\ \det \mathcal{A}(t, \mu) \det H_{22}^{-1}(t, 0) H_{22}^{-1}(t, 1), & \text{for } t \in \Gamma \text{ and } t \neq \alpha_k, \beta_k, \\ \det H_{22}(\beta_k, \mu) H_{22}^{-1}(\beta_k, 1), & \text{for } t = \beta_k \end{cases}$$

is continuous on $\mathfrak{M}$, and

$$\text{ind } A = -\frac{1}{2\pi} [\arg f_A(t, \mu)]_{\mathfrak{M}}. \quad (7)$$

Let us explain the meaning of the right-hand side of formula (7). If the operator has the form

$$A = \sum_{j=1}^k \prod_{l=1}^m (F_{jl}P + G_{jl}Q) \quad (F_{jl}, G_{jl} \in \Lambda_n), \quad (8)$$

then the set of values of the function $f_A(t, \mu)$ consists of a finite number of closed continuous curves, which are naturally oriented: at the points of continuity of all the matrix-functions $F_{jl}(t)$ and $G_{jl}(t)$, motion along the curve $f_A(t, \mu)$ is determined by the change of t along the contour in the positive direction, while along the additional arcs it is determined by the change of μ from 0 to 1. The number $[\arg f_A(t, \mu)]_{\mathfrak{M}}/2\pi$ is equal to the number of turns of the curve $f_A(t, \mu)$ around the origin.

Let $A \in \mathfrak{A}_n$. Then the function $f_A(t, \mu)$ is the uniform limit of a sequence of functions $f_{A_N}(t, \mu)$, where the A_N are operators of the form (8). For sufficiently large N , the number $[\arg f_{A_N}(t, \mu)]_{\mathfrak{M}}$ does not depend on N , and by definition

$$[\arg f_A(t, \mu)]_{\mathfrak{M}} = \lim_{N \rightarrow \infty} [\arg f_{A_N}(t, \mu)]_{\mathfrak{M}}.$$

The proof of Theorem 4 is carried out according to the following scheme. Since both sides of formula (7) are continuous functions of the operator A , it is sufficient to establish this formula for operators of the form (9). Let A be an operator of the form (9), and let $L = \Xi(F_{jl}P + G_{jl}Q)$ be its linear extension (see (2)). The operator L is a singular integral operator with matrix coefficients in the space $L_2^r(\Gamma)$, where $r = (mk - k + 1)n$. Let $\tilde{F}_{jl}(t)$ and $\tilde{G}_{jl}(t)$ ($t \in \tilde{\Gamma}$) be, respectively, certain matrix-functions coinciding with $F_{jl}(t)$ and $G_{jl}(t)$ on the contour Γ and continuous on the closed set $\tilde{\Gamma} \setminus \Gamma$; then $\text{ind } A = \text{ind}(B_r \tilde{L} B_r + C_r)$, where $\tilde{L} = \Xi(\tilde{F}_{jl}P + \tilde{G}_{jl}Q)$. Thus, the problem of computing the index of the operator A is reduced to the problem of computing the index of the operator $\tilde{A} = B_r \tilde{L} B_r + C_r$, acting in $L_2^r(\tilde{\Gamma})$, which was solved in (1). From the formula for the index of the operator $\tilde{A}$ given in (1), formula (8) is easily derived.

5. Let $t_1, \dots, t_r$ ($r = 0, 1, \dots$) be fixed points on the contour Γ , distinct from the endpoints of the open arcs. By $\mathfrak{A}_n(t_1, \dots, t_r)$ we denote the algebra obtained by closing in the algebra $\mathfrak{A}_n$ the set of operators of the form (8) for which the matrix-functions $F_{jl}(t)$ and $G_{jl}(t)$ are continuous at all points of the contour Γ , except possibly the points $t_1, \dots, t_r$. In this case the factor algebra $\mathfrak{A}(t_1, \dots, t_r)/\mathfrak{S}_\infty$ is isomorphic and isometric to the algebra of matrix-functions defined on the contour obtained from the contour Γ by splitting each point t_k ($k = 1, \dots, r$) into t_k^- and t_k^+ and adding intervals with endpoints t_k^- and t_k^+ , and also by adding intervals to each of the endpoints α_k and β_k of the open arcs.

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