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Abstract

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MATHEMATICS

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ON THE NECESSITY OF A FREQUENCY CONDITION FOR THE ABSOLUTE STABILITY OF STATIONARY SYSTEMS IN THE CRITICAL CASE OF A PAIR OF PURELY IMAGINARY ROOTS

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At the present time there are many works in which sufficient frequency conditions for absolute stability ⁽¹⁾ of nonlinear controlled systems have been obtained. As indicated in the survey ⁽²⁾, one of the main problems is the question of the necessity of these conditions for absolute stability. In the present article a class of systems is singled out for which the frequency criterion obtained by V. A. Yakubovich ⁽³⁾ is not only a sufficient but also a necessary condition for absolute stability.

Consider the system

$$dx/dt = Ax + q\varphi(\sigma), \quad \sigma = (r, x), \quad (1)$$

where q, r are constant n -dimensional vectors, A is a constant $n \times n$ matrix, and $\varphi(\sigma)$ is a continuous scalar function. The matrix A has two purely imaginary eigenvalues $\pm i\omega_0$ ($\omega_0 > 0$); the remaining eigenvalues (if $n > 2$) have negative real parts.

Introduce the transfer function $\chi(\lambda) = (r, (A - \lambda I)^{-1}q)$, where λ is a complex variable, I is the identity $n \times n$ matrix, and introduce the notation

$$\alpha = \frac{1}{\omega_0} \lim_{\omega \rightarrow \omega_0} (\omega_0^2 - \omega^2) \operatorname{Im} \chi(i\omega), \quad \beta = \lim_{\omega \rightarrow \omega_0} (\omega_0^2 - \omega^2) \operatorname{Re} \chi(i\omega),$$

$$\pi(\omega) = \operatorname{Re} \left[\left(1 + \frac{i\omega\beta}{\alpha\omega_0^2} \right) \chi(i\omega) \right], \quad \pi_\infty = \lim_{\omega \rightarrow \infty} \pi(\omega),$$

$$\varkappa = - \lim_{\lambda \rightarrow \infty} \lambda \chi(\lambda).$$

In this notation ω is a real variable. The function $\pi(\omega)$ is extended to the value ω_0 by continuity.

Suppose that the following conditions are satisfied:

$$0 < \varphi(\sigma)/\sigma \leq \mu \quad \text{for } \sigma \neq 0, \quad (2)$$

$$\alpha > 0, \quad \beta\chi > 0, \quad (3)$$

$$\lim_{\omega \rightarrow \infty} \omega^2(\pi(\omega) - \pi_\infty) > 0, \quad \pi(\omega) > \pi_\infty \quad \text{for all } \omega \geq 0. \quad (4)$$

Theorem 1. *Let conditions (3), (4) be satisfied. Then, for absolute stability of system (1) in the class of nonlinearities satisfying condition (2), it is necessary and sufficient that the frequency condition*

$$\pi(\omega) + 1/\mu > 0 \quad (5)$$

be satisfied for all $\omega \geq 0$.

Theorem 2. *Let condition (3) be satisfied. Then, for absolute stability of system (1) in the class of nonlinearities satisfying condition (2), it is necessary that the inequality $\mu \leq \alpha\omega_0^2/\beta\chi$ be satisfied.*

We note that the result formulated in Theorem 2, for the case $n = 3$, is contained in the monograph of V. A. Pliss ⁽⁴⁾.

Let the numbers $h, H, \delta, \varepsilon$ satisfy the inequalities $H \geq h > \alpha\omega_0^2/\beta\chi$, $0 < \delta < \varepsilon < \delta + \delta^4$. Denote by $E(h, H, \delta)$ the class of functions $\varphi(\sigma)$ satisfying the following conditions:

$$h\sigma^2 \leq \varphi(\sigma)\sigma \leq H\sigma^2 \quad \text{for } |\sigma| \leq \delta,$$

$$0 < \varphi(\sigma)\sigma \leq H\sigma^2 \quad \text{for } \delta \leq |\sigma| \leq \varepsilon, \quad \varphi(\sigma) = \delta^4 \quad \text{for } \sigma \geq \varepsilon,$$

$$\varphi(\sigma) = -\delta^4 \quad \text{for } \sigma \leq -\varepsilon.$$

Theorem 3. *Suppose conditions (3) are fulfilled and $\varphi(\sigma) \in E(h, H, \delta)$, where δ is sufficiently small in comparison with the norms $\|A\|$, $\|q\|$, $\|r\|$. Then there exists a trajectory of system (1) which does not tend to the origin as $t \rightarrow +\infty$. If, in addition, condition (4) is fulfilled, then system (1) has a periodic solution distinct from the equilibrium state.*

Corollary 1. *Theorem 1 is valid if inequality (2) is replaced by the condition $\varepsilon_0 \leq \varphi(\sigma)/\sigma \leq \mu$, where ε_0 is a sufficiently small positive number.*

Corollary 2. *For every $n > 2$ there exists a completely controllable and completely observable system of the form (1) with a nonlinearity satisfying the generalized Hurwitz conditions and possessing a periodic solution distinct from the equilibrium state.*

For $n = 3$ this fact was established by V. A. Pliss ⁽⁴⁾.

In proving the theorems stated, without loss of generality we shall assume that the matrix A has the form

$$A = \begin{pmatrix} 0 & -\omega_0 & 0 \\ \omega_0 & 0 & 0 \\ 0 & 0 & B \end{pmatrix},$$

where B is a Hurwitz matrix such that the system $dy/dt = By$ has the Lyapunov function** $W(y) = \sum_{i=1}^{n-2} y_i^2$. Then $x = (r, q)$, $\alpha = -(r_1 q_1 + r_2 q_2)$,

$$\beta = (r_1 q_2 - r_2 q_1) \omega_0.$$

For definiteness we shall assume that $r_2 \neq 0$ (for $r_1 \neq 0$ the reasoning is analogous).

Introduce the notation:

$$\gamma = r_1^2 + r_2^2, \quad V(x) = (x_1 q_1 + x_2 q_2) + \alpha(r, x)/\gamma,$$

$$U(x) = (A^* r, x), \quad w(x) = [x_1^2 + x_2^2]/2,$$

$$\Omega_1 = \left\{ x \mid (r, x) = 0, \sum_{i=3}^n x_i^2 < \delta^3, w(x) > 1 \right\},$$

$$\Omega_2 = \{x \mid x \in \Omega_1, r_2 x_1 > 0\}.$$

Let $p \in \Omega_2$; $x(t, p)$ be the solution of system (1) on the time interval $[0, +\infty)$ with initial data $x(0, p) = p$; $t_{i,p}$ ($i = 1, 2$) be the i -th instant

* M. A. Aizerman and F. R. Gantmakher showed ⁽¹⁾ that the frequency criterion of V. A. Yakubovich (5), established for the case of two purely imaginary roots, formally coincides with the frequency condition of V. M. Popov (established for the case when A is a Hurwitz matrix). Namely, in ⁽¹⁾ it was shown that

in this case the variable parameter in V. M. Popov's criterion is uniquely determined, and V. M. Popov's criterion takes the form (5). Analogously one can show that condition (5) is a necessary and sufficient condition for the existence of a parameter in V. M. Popov's criterion for the class of nonlinearities $\varepsilon_0 \leq \varphi(\sigma)/\sigma \leq \mu$, where ε_0 is a sufficiently small number, if conditions (4) are fulfilled. Therefore Corollary 1 identifies classes of systems (unfortunately, not effectively) for which the frequency condition of V. M. Popov is not only a sufficient but also a necessary condition for absolute stability. In this case there are also singled out classes of systems for which the Hurwitz sector does not coincide with the sector of absolute stability.

** If a vector is denoted by the symbol a , then by the symbol a_i is meant its i -th component. In particular, x_i, y_i, z_i, p_i denote the components of the vectors x, y, z, p .

of intersection of the trajectory $x(t, p)$ with the hypersurface $\sigma = 0$; $\tau_{i,p}$ ($i = 1, 2$) is the i -th moment of intersection of the trajectory $x(t, p)$ with the hypersurface $\sigma = \varepsilon$. By direct computation one can verify the validity of the following assertions.

Lemma 1. $dV(x)/dt = O(\delta^{3/2})$, $dU(x)/dt = O(\delta^{3/2})$ for $x \in \Omega_1$.

Lemma 2. For sufficiently small δ , on the time intervals $[0, \tau_{1,p}]$ and $[\tau_{2,p}, t_{1,p}]$ the solution $x(t, p)$ may be regarded as a function of σ .

Lemma 3. $\tau_{1,p} = O(\delta)$, $t_{1,p} - \tau_{2,p} = O(\delta)$.

Lemma 4. For sufficiently small δ , on the time interval $[\pi/2\omega_0, t_{2,p}]$ the estimate

$$\sum_{i=3}^n x_i^2(t, p) < \delta^3$$

is valid.

Lemma 5. $dV(x)/d\sigma = O(\delta^{3/2})$, $dU(x)/d\sigma = O(\delta^{3/2})$ for $x \in \Omega_1$.

Lemma 6. On the time interval $[0, \tau_{1,p}]$ the estimate

$$V(x(\sigma, p)) = V(p) + O(\delta^{3/2}), \quad U(x(\sigma, p)) = U(p) + O(\delta^{3/2})$$

is valid.

On the time interval $[\tau_{2,p}, t_{1,p}]$ the estimate

$$V(x(\sigma, p)) = -V(p) + O(\delta^{3/2}), \quad U(x(\sigma, p)) = -U(p) + O(\delta^{3/2})$$

is valid.

Lemma 7.

$$V(p) = -\beta z_1/r_2 \omega_0 + O(\delta^{3/2}), \quad U(p) = \gamma z_1 \omega_0/r_2 + O(\delta^{3/2}),$$

where z is some element of the set $\{z \mid z \in \Omega_2, z_3 = \dots = z_n = 0\}$.

Proof of Theorem 3. In the proof of the theorem we use some ideas of V. A. Pliss, set forth in [4].

To the point $p \in \Omega_2$ assign the point $x(t_{2,p}, p)$. Denote the transformation of Ω_2 thus obtained by T . We shall prove that $T\Omega_2 \subset \Omega_2$. From Lemma 4 and the conditions defining the class $E(h, H, \delta)$, it follows that, in order to prove this inclusion, it is enough to show that $w(p) < w(x(t_{2,p}, p))$ for $|p_1| \leq N$, where N is a number sufficiently large in comparison with the norms $\|A\|, \|q\|, \|r\|$.

First we estimate the increment of the function w along the trajectory $x(t, p)$ as the latter passes through the strip $0 \leq \sigma \leq \varepsilon$. For this purpose we shall consider the function w on the trajectory $x(t, p)$, for $t \in [0, \tau_{1,p}]$ and for $t \in [\tau_{2,p}, t_{1,p}]$, as a function of σ . In doing so an ambiguity arises, since the trajectory $x(t, p)$ passes through the strip $0 \leq \sigma \leq \varepsilon$ twice. To avoid this ambiguity, on the interval $[0, \tau_{1,p}]$ we shall attach to w the plus sign, and on $[\tau_{2,p}, t_{1,p}]$ the minus sign.

Consider the equality

$$dw/d\sigma = [x_1 q_1 + x_2 q_2] \varphi(\sigma) / (A^* r, x) + (r, q) \varphi(\sigma).$$

From this formula and the estimates of Lemma 6, we find

$$\frac{dw_+}{d\sigma} - \frac{dw_-}{d\sigma} = 2\varphi(\sigma) \frac{\alpha U(p)\sigma/\gamma + \chi V(p)\varphi(\sigma) + O(\delta^{3/2})}{[\chi\varphi(\sigma) + O(\delta^{3/2})]^2 - [U(p)]^2}.$$

Using Lemma 7, we obtain

$$\frac{dw_+}{d\sigma} - \frac{dw_-}{d\sigma} = 2\varphi(\sigma) \frac{r_2 z_1 [-\alpha \omega_0^2 + \beta \chi \varphi(\sigma)] + O(\delta^{3/2})}{\omega_0^3 \gamma^2 z_1^2 + O(\delta^{3/2})}.$$

From the inequality $w(z) > 1$ follows the relation $z_1^2 > r_2^2/\gamma$. Hence we obtain the estimate

$$\begin{aligned} & w(x(t_{1,p}, p)) - w(x(\tau_{2,p}, p)) + w(x(\tau_{1,p}, p)) - w(p) \\ &= \int_0^\delta [dw_+/d\sigma - dw_-/d\sigma] d\sigma + \int_\delta^\varepsilon [dw_+/d\sigma - dw_-/d\sigma] d\sigma \\ &> [-\alpha \omega_0^2 + \beta \chi h] \delta^3 r_2 h / 3 z_1 \gamma^2 \omega_0^3 + O(\delta^4). \end{aligned}$$

From the conditions defining the class $E(h, \bar{H}, \delta)$, it follows that $w(x(\tau_{2,p}, p)) - w(x(\tau_{1,p}, p)) = O(\delta^4)$. Hence $w(x(\tau_{1,p}, p)) - w(p) > 0$. Continuing the same reasoning further for $t \in [t_{1,p}, t_{2,p}]$, we prove the inequality $w(x(t_{2,p}, p)) - w(p) > 0$.

From the inclusion $T\Omega_2 \subset \Omega_2$ there follows the existence of a solution not tending to zero as $t \rightarrow +\infty$.

Let us prove the existence of a periodic solution. From the results of V. A. Yakubovich ⁽³⁾, p. 610, it follows that, when conditions (3), (4) are fulfilled, there exists such a symmetric positive definite $n \times n$ matrix M that the derivative, along the system (1), of the function

$$v(x) = x^* M x + (\beta / \alpha \omega_0^2) \int_0^\sigma \varphi(\sigma) d\sigma - \sigma^2 / 2\kappa$$

satisfies the estimate

$$dv(x)/dt \leq -[\sigma - (\beta\kappa / \alpha \omega_0^2) \varphi(\sigma)] \varphi(\sigma).$$

Choose the number δ so small that the inequality $\varepsilon > 2\delta^2\kappa\beta / \alpha \omega_0^2$ is fulfilled, and the number v_0 so large that, for $v(p) = v_0$, the estimate $H^2\varepsilon(\tau_{1,p} + t_{1,p} - \tau_{2,p})\beta\kappa / \alpha \omega_0^2 < \delta^4(\tau_{2,p} - \tau_{1,p})/2$ is fulfilled.

From the conditions defining the class $E(h, \bar{H}, \delta)$, it follows that on the trajectory $x(t, p)$, for $t \in [0, \tau_{1,p}]$ and for $t \in [\tau_{2,p}, t_{1,p}]$, the inequality $dv(x)/dt < H^2\varepsilon^2\beta\kappa / \alpha \omega_0^2$ holds, while for $t \in [\tau_{1,p}, \tau_{2,p}]$ the estimate $dv(x)/dt < -\delta^4\varepsilon/2$ is valid. Hence it follows that

$$v(x(t_{1,p}, p)) - v(p) < -\delta^4\varepsilon(\tau_{2,p} - \tau_{1,p})/2 + H^2\varepsilon^2(\tau_{1,p} + t_{1,p} - \tau_{2,p})\beta\kappa / \alpha \omega_0^2 < 0.$$

Continuing the same reasoning further for $t \in [t_{1,p}, t_{2,p}]$, we prove the inequality $v(x(t_{2,p}, p)) - v(p) < 0$. From the latter inequality it follows that $T\bar{Q} \subset Q$, where $Q = \{x \mid x \in \Omega_2, v(x) < v_0\}$. Hence, by the well-known Brouwer theorem, in the domain Q there exists a point fixed with respect to the transformation T . This proves the existence of a periodic motion of system (1).

Theorem 2 follows directly from Theorem 3. Indeed, if $\mu > \alpha \omega_0^2 / \beta\kappa$, then there exists a function $\varphi(\sigma) \in E(h, \bar{H}, \delta)$ satisfying condition (2) (one must take the numbers h and $\bar{H}$ so that $\alpha \omega_0^2 / \beta\kappa < h \leq \bar{H} < \mu$).

The necessity of condition (5) in Theorem 1 follows from Theorem 2. Indeed, suppose that at some point ω^* the inequality $\pi(\omega^*) + 1/\mu \leq 0$ holds. Then from conditions (4) we obtain $\pi_\infty + 1/\mu < 0$. But $\pi_\infty = -\beta\kappa / \alpha \omega_0^2$. Hence $\mu > \alpha \omega_0^2 / \beta\kappa$. The sufficiency of condition (5) follows from the results of V. A. Yakubovich published in ⁽³⁾ (corollary to Theorem 2).

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* By $\bar{Q}$ we denote the closure of Q .

Note: Figure translations are in progress. See original paper for figures.

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