

ON PERIODIC GROUPS WITH CERTAIN FINITENESS CONDITIONS

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Abstract

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MATHEMATICS

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ON PERIODIC GROUPS WITH CERTAIN FINITENESS CONDITIONS

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In accordance with the work ⁽¹⁾, a locally finite group with extremal Sylow p -subgroups (for a given prime number p) is called an S_{pF} -group. In particular, if a locally finite group is an S_{pF} -group for every prime divisor p of the orders of its elements, then it is called an SF -group.

In the present paper a result is formulated which gives a necessary and sufficient condition for the invariance of a maximal complete abelian 2-subgroup in an S_2F -group with an infinite Sylow 2-subgroup (see Theorem 1). Using Theorem 1, it is easy to prove that an arbitrary SF -group is an extension of an abelian group by means of an SF -group with finite Sylow 2-subgroups (see Theorem 2). Hence, and from Theorems 3, 4 and a result of S. N. Chernikov ⁽²⁾, it follows that questions 2.1, 8.1, 8.2 posed in the survey ⁽³⁾, as well as question 23 from ⁽⁴⁾, need only be considered for SF -groups with finite Sylow 2-subgroups; and since locally soluble SF -groups have been studied up to SF -groups with finite Sylow p -subgroups ⁽⁵⁾, and locally finite groups of odd order are locally soluble ⁽⁶⁾, the significance of Theorems 2, 3, 4 in connection with the consideration of the indicated questions becomes clear.

With the aid of Theorems 2, 3 the author proves that, for locally finite groups, the minimality condition for abelian subgroups implies the minimality condition for subgroups, and that Sylow p -subgroups in an SF -group are conjugate. Consequently, questions 2.1, 8.2 from ⁽³⁾ are equivalent.

The conjugacy of Sylow p -subgroups in locally finite groups with the minimality condition for subgroups was first established in the work ⁽⁷⁾.

In Section 2 a class of p -groups is singled out, called by the author biprimatively finite; for $p = 2$ these coincide with 2-groups, and to this class of groups Blackburn's result on Chernikov p -groups ⁽⁸⁾ is generalized (see Theorem 8 of the present work).

An example of a group constructed by P. S. Novikov and S. I. Adian ⁽⁹⁾ shows that Theorem 8 cannot be generalized to arbitrary p -groups, even under the condition of finiteness of all abelian subgroups in the group.

At the same time, it follows from a result of E. S. Golod ⁽¹⁰⁾ that a biprimatively finite p -group need not be locally finite.

As a corollary of Theorem 8 we obtain that a biprimatively finite p -group (and for $p = 2$, an arbitrary 2-group) satisfying the minimality condition for abelian subgroups is a Chernikov p -group (see Corollary 1).

Corollary 1 for $p = 2$ was stated without proof by S. P. Strunkov in the abstracts of the Batumi Symposium, 1967, and for p -groups in which any two elements generate a finite group it was proved by V. G. Viljaper ⁽¹¹⁾.

1. A group containing (not containing) involutions will be called a group of even (odd) order.

$C_G(M)$ is the centralizer of the set M of elements of the group G .

$N_G(A)$ is the normalizer of the subgroup A in the group G .

$\pi(G)$ is the set of prime divisors of the orders of elements of the group G .

If A and B are subsets of the group G , then AB is the set of elements of G of the form $g = ab$ ($a \in A$, $b \in B$).

$A\lambda B$ is the semidirect product of the groups A and B , where A is a normal divisor in $A\lambda B$.

An element g of the group G is called strictly real with respect to the involution i from G if $igi = g^{-1}$.

By the rank of a group we shall mean the special rank in the sense of Mal'cev ⁽¹²⁾.

Definition 1. A Sylow p -subgroup P of a countable locally finite group G is called **projective** in G if the group G has a chain of finite subgroups

$$N_1 \subset N_2 \subset \dots \subset N_s \subset \dots,$$

whose union coincides with G , such that $P \cap N_s$ ($s = 1, 2, \dots$) is a Sylow p -subgroup in N_s .

Every countable locally finite group G , for any $p \in \pi(G)$, has a projective Sylow p -subgroup. This is not hard to prove by using the well-known method of projections ⁽¹³⁾.

Definition 2. We shall say that an S_{pF} -group has a p -complete part if a maximal complete abelian p -subgroup in it is invariant. In particular, if the Sylow p -subgroups of an S_{pF} -group are finite, then we shall agree to regard the identity subgroup as the p -complete part.

If G is an S_{pF} -group, then an element g of order p from G will be called an element of the first kind if $C_G(g)$ has a p -complete part, and an element of the second kind otherwise.

Definition 3. The rank of the p -complete part of some projective Sylow p -subgroup of a countable S_{pF} -group G will be denoted by $t_p(G)$ and will be called the p -complete rank of the group G . The index of the p -complete part in a subgroup P of the group G will be called the p -complete index of the group G and will be denoted by $l_p(G)$.

If the Sylow p -subgroups of the S_{pF} -group G are finite (in particular, if G contains no elements of order p), then we shall agree to set $t_p(G) = 0$.

It is easy to prove that the p -complete rank of a countable S_{pF} -group G does not depend on the choice of the projective Sylow p -subgroup, and if for some subgroup H of G $t_p(H) = t_p(G)$, then $l_p(H) \leq l_p(G)$, and also the p -complete rank of a subgroup does not exceed the p -complete rank of the group.

Theorem 1. *Let G be an S_2F -group with an infinite Sylow 2-subgroup, satisfying the following condition:*

) if H is a subgroup of odd order and $N_G(H)$ contains a quasicyclic 2-subgroup Q , then $Q \subset C_G(H)$. Then a maximal complete abelian 2-subgroup of G is invariant in G .*

Theorem 1 is proved by contradiction, i.e., it is assumed that there exist S_2F -groups (without impairing the generality of the argument, they may be assumed countable) satisfying condition *), but not having 2-complete parts.

In the class of all such countable groups one chooses the subset of groups of least 2-complete rank, and in this subset the class of all groups of least 2-complete index; this class is denoted by K_2 .

In the first part of the proof of Theorem 1 it is established that in K_2 there exists a group in which every involution is an element of the first kind. Here the main tool of the proof is Glauberman's theorem ⁽¹⁴⁾ (and if one does not use this theorem, then Grün's theorem ⁽¹⁵⁾).

In the second part of the proof of Theorem 1, instead of K_2 one considers the subset K'_2 of all groups from K_2 in which every involution is an element of the first kind.

It is proved that K'_2 has a group such that it is either isomorphic to a group of type $PSL(2, T)$ over a field T of odd characteristic, or has-

gives a subgroup, some infinite factor group of which is isomorphic to one of the groups: $SL(2, Q)$, the Suzuki group $S(Q)$, or the unitary group $U_3(Q)$, where Q is a field of even characteristic.

In the first case we obtain a contradiction with condition *, since in a group of type $PSL(2, T)$ with an infinite Sylow 2-subgroup and over a field of odd characteristic there exists an abelian subgroup B of odd order, and in its normalizer there is a quasicyclic 2-subgroup which does not centralize the subgroup B .

In the second case we obtain a contradiction with the extremality of the Sylow 2-subgroup in an S_2F -group, since in infinite groups of type $SL(2, Q)$, $S(Q)$,

$U_3(Q)$ over a field of even characteristic, the Sylow 2-subgroups are not extremal.

The main tools in the second part of the proof of Theorem 1 are the result of Gorenstein and Walter ⁽¹⁶⁾, the result of Suzuki ⁽¹⁷⁾ on finite groups with an independent Sylow 2-subgroup, Theorem 14.3.1 of ⁽¹⁵⁾, and the Brauer–Suzuki result ⁽¹⁸⁾.

Theorem 2. *An SF-group is an extension of an abelian group by an SF-group with finite Sylow 2-subgroups.*

Theorem 3. *A locally finite group with the minimality condition for (abelian) subgroups is an extension of an abelian group by a group with finite Sylow 2-subgroups and the minimality condition for (abelian) subgroups.*

Theorem 4. *If an infinite simple locally finite group of finite rank exists, then its Sylow 2-subgroups are finite.*

Theorem 5. *For locally finite groups, the minimality condition for abelian subgroups implies the minimality condition for subgroups.*

The question of the validity of Theorem 5 for arbitrary periodic groups remains open.

In the proof of Theorem 5, as well as of Theorem 7, together with Theorem 3 the following lemma is used.

Lemma. *Let G be a finite group, H a proper subgroup of even order coinciding with its normalizer, $j \in G$ an involution and $j \notin H$, $D(j)$ the subgroup of H generated by all elements which are strictly real with respect to j , and i some involution from H .*

If the subgroup H is 2-transversal to its conjugates, i.e. intersects any other subgroup conjugate to it in a subgroup of odd order, then

- 1) *the subgroup $D(j)$ is of odd order, and for every strictly real element $a \neq 1$ of H with respect to j , the centralizer $C_G(a)$ is of odd order;*
- 2) *in each adjacent class $hC_H(i)$ ($h \in H$) there exists one and only one element from $D(j)$ which is strictly real with respect to j , and $H = D(j)C_H(i)$;*
- 3) *no Sylow p -subgroup of $D(j)$ is centralized by the involution j ;*
- 4) *if $D = H \cap jHj$, then $D\lambda\langle j \rangle$.*

Theorem 6. *If in a locally finite group the Sylow 2-subgroups are extremal and the subgroups of odd order are finite, then the group itself is extremal.*

Theorem 7. *The Sylow p -subgroups of an SF-group are conjugate.*

2. Let G be a periodic group, and p some prime number, satisfying the following condition:

If H is a subgroup of G , N its invariant extremal subgroup, then in the factor group H/N any two elements of order p (if H/N has such elements) generate a finite group.

In this case we shall call the group G **biprimatively finite relative to p** . If G is biprimatively finite relative to every $p \in \pi(G)$, then we shall call the group G **biprimatively finite**.

Every periodic group of odd order is biprimatively finite with respect to 2.

On the other hand, as the example of E. S. Golod⁽¹⁰⁾ shows, there exist biprimatively finite p -groups (for any p) that are not locally finite groups.

Theorem 8. If a biprimatively finite p -group (and for $p = 2$ an arbitrary 2-group) possesses a finite maximal elementary abelian subgroup, then the group is extremal.

Corollary 1. If a biprimatively finite p -group (and for $p = 2$ an arbitrary 2-group) satisfies the minimality condition for abelian subgroups, then the group is extremal.

Corollary 2. A biprimatively finite p -group (and for $p = 2$ an arbitrary 2-group) is finite if and only if it is generated by a finite number of generators and some maximal elementary abelian subgroup of it is finite.

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