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Abstract

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MATHEMATICS

G. L. GARG (INDIA)

AN ANALOGUE OF THE KURATOWSKI-DUGUNDJI THEOREM FOR THE CATEGORY OF METRIZABLE UNIFORM SPACES WITH UNIFORMLY CONTINUOUS MAPS

(Presented by Academician P. S. Aleksandrov on 23 IV 1970)

The well-known theorem of K. Kuratowski*, generalized by Dugundji ^(2,3) to the category of metrizable topological spaces, connects the question of the continuous extension $ef : X \rightarrow Y$ of a continuous mapping $f : A \rightarrow Y$ of a closed set A (under appropriate dimensional restrictions imposed on $X \setminus A$) with the properties of connectedness and local connectedness of the space Y in higher dimensions, and also with the problem of retracting an arbitrary space Z onto the image iY of the space Y under a closed** topological embedding i (and, of course, under analogous restrictions imposed on $Z \setminus iY$).

The main purpose of this work is to study the question of extending mappings in the category of metrizable uniform spaces with uniformly continuous mappings, and also to obtain an analogue of the already mentioned Kuratowski theorem for this case.

As it turns out, in these questions the following dimensional characteristic*** plays an essential role: the **relative large dimension** $r\Delta d(M)$ of a set M of a uniform space X . By definition

$$r\Delta d(M) = \sup \Delta d(B) / B\delta(X \setminus M),$$

where Δd is the large dimension in the sense of Isbell ⁽⁴⁾, and δ is the relation of nearness generated in the known way by the uniform structure of the space X . If X is metrizable (uniformly), and ρ is one of its uniform metrics, then the relation $B\delta C$ is equivalent to the equality $\rho(B, C) = 0$, and, consequently,

$$r\Delta d(M) = \sup_B \Delta d(B) / \rho(B, X \setminus M) > 0.$$

Since the dimension Δd is monotone, $r\Delta d(M) \leq \Delta d(M)$.

Here are the assertions that appear to us to be the principal ones:

Theorem 1. For a complete space**** Y the following three conditions are equivalent:

- I. The space Y is locally connected in all dimensions***** $< n$ (and connected in all dimensions***** $< n$);
- II. If A is closed in a space X , while $X \setminus A$ is precompact and $r\Delta d(X \setminus A) \leq n$, then every mapping $f : A \rightarrow Y$ is extendable to a mapping $f_U : U \rightarrow Y$ (respectively $f_X : X \rightarrow Y$), where U is some uniform neighborhood of the set A ;

* See ⁽¹⁾, § 53, IV, or ⁽²⁾, Ch. 3, § 9.

** This is, obviously, equivalent to the condition of closedness of the image iY in the space Z , usually asserted in this situation.

*** Proposed by Yu. M. Smirnov.

**** By a “space” we shall everywhere below mean a metrizable uniform space; by a “mapping,” a uniformly continuous mapping; by an “extension of a mapping,” a uniformly continuous extension; by a “retraction,” a uniformly continuous retraction; finally, by an “embedding,” a uniform embedding.

***** In the sense of Aleksandrov-Lefschetz, see ⁽²⁾, Ch. I, § 17, or ⁽¹⁾, § 53, IV.

- III. For every such closed embedding $i : Y \rightarrow Z$ in a space Z , such that $Z \setminus iY$ is precompact and $r\Delta d(Z \setminus iY) \leq n$, there exists a retraction $r_U : U \rightarrow iY$ (respectively $r_Z : Z \rightarrow iY$), where U is some uniform neighborhood of the set iY^* .

Theorem 1’. The assertions of Theorem 1 remain true if in condition II the dimension $r\Delta d(X \setminus A)$ is replaced by $\Delta d(X \setminus A)$, if in the same condition it is required that the space X be compact**, and, finally, if the replacement indicated above and the requirement are combined.

Theorem 1’’. If Y is complete and $\Delta dY \leq n$, then each of the conditions of Theorem 1 is equivalent to the condition obtained from condition III by replacing the dimension $r\Delta d(Z \setminus iY)$ by $\Delta d(Z \setminus iY)$.

We give a brief proof of the equivalence of the “integral” conditions I and II. Since $r\Delta d(X \setminus A) \leq \Delta d(X \setminus A)$, the weakest of the conditions of Theorem 1’ and condition II will be condition II, obtained simultaneously by replacing the dimension $r\Delta d(X \setminus A)$ by $\Delta d(X \setminus A)$ and by requiring compactness of the space X . Apply it to the case of the cube $X = I^n$ and a continuous mapping $f : A \rightarrow Y$, where A is closed in X . Since this mapping f is uniformly continuous and $\Delta d(X \setminus A) \leq \Delta dX = \dim X = n$, this is possible. Hence every such mapping $f : A \rightarrow Y$ extends to a continuous mapping $ef : X \rightarrow Y$. But then, by Kuratowski’s theorem, condition I is fulfilled. Let us now derive condition II from condition I. For this purpose consider the completion cX of the space X mentioned in II (it is metrizable). Denote by $\overline{M}$ the closure of a set M ($M \subseteq X$) in the completion cX . It is not hard to see that $\overline{X \setminus A}$

is compact, while $\bar{A}$ is a completion of the set A . Therefore the mapping f extends to a mapping $\bar{f} : \bar{A} \rightarrow Y$ (since Y is complete). The intersection $B = \bar{A} \cap (\overline{X \setminus A})$ is compact and lies in $\overline{X \setminus A}$. Since $\overline{X \setminus A} \setminus B$ is open, it is the union of a countable number of compacta C_j (where $C_j \subseteq C_{j+1}$), for which $\dim C_j = \Delta d C_j \leq r \Delta d(X \setminus A) \leq n$. Hence $\dim(\overline{X \setminus A} \setminus B) \leq n$. But then, by condition I, the mapping $\bar{f}_B : B \rightarrow Y$, according to the classical Kuratowski theorem, can be extended to a mapping $\tilde{f} : \overline{X \setminus A} \rightarrow Y$. Since $\overline{X \setminus A}$ is compact, and on B the mappings $\bar{f}$ and $\tilde{f}$ coincide, the final mapping $cf : cX \rightarrow Y$ defined by them is uniformly continuous. This proves not only the first part of Theorem 1, but also all of Theorem 1'.

An essential role in the proof of the remaining part of Theorem 1 is played by the following theorem, analogous to a well-known proposition of Hanner ⁽⁵⁻⁷⁾.

Theorem 2. For every mapping $f : A \rightarrow Y$ of a set A , closed in the space X , into the space Y , there exist an n -space Z , a mapping $F : X \rightarrow Z$, and a closed uniform embedding $i : Y \rightarrow Z$, satisfying the following conditions:

$$\alpha) \quad i(f(x)) = F(x), \quad \text{if } x \in A,$$

$$\beta) \quad \text{on the set } X \setminus A \text{ the mapping } F \text{ is a homeomorphism } ** ,$$

$$\gamma) \quad \text{on every set } B \text{ far from } A****, \text{ the mapping } F \text{ is a uniform homeomorphism.}$$

The proof of this theorem is constructive: on the set $Z = Y \cup (X \setminus A)$ (assuming that $Y \cap (X \setminus A) = \emptyset$) a metrizable uniform structure is constructed which turns the set Z into the desired space. From this theorem we immediately obtain the important general

Corollary. For any space Y the following two conditions are equivalent:

* In the "integral" conditions II and III obtained with the aid of the changes indicated in parentheses, explanations about neighborhoods U are not needed.

** Then the difference $X \setminus A$ will automatically be precompact.

*** This homeomorphism is uniformly continuous (since F is uniformly continuous). But it need not be uniform.

**** I.e., $B\delta A$ or, equivalently, $\rho(B, A) > 0$.

- II. If A is closed in the space X , then every mapping $f : A \rightarrow Y$ can be extended to some uniform neighborhood U of the set A (to all of X);
- III. For every closed embedding $i : Y \rightarrow Z$ there exists a retraction $r_U : U \rightarrow iY$ (respectively $r_Z : Z \rightarrow iY$), where U is some uniform neighborhood of the set iY .*

Now, in order to prove the remaining part of Theorem 1, it must be observed that, under the hypotheses of Theorem 2, from the precompactness of the difference $X \setminus A$ there follows the precompactness of the difference $Z \setminus iY$, and that $r\Delta d(Z \setminus iY) = r\Delta d(X \setminus A)$ (by condition γ).** Let us also note that Theorem 6 (Isbell ⁽⁴⁾, Ch. V) in essence asserts that, for any uniform space X and its subspace A , the equality

$$\Delta dX = \text{Max}\{\Delta dA, r\Delta d(X \setminus A)\}$$

holds. Hence, from Theorem 2 it also follows that the assertion of our corollary remains true for every space Y of dimension $\Delta dY \leq n$, if in condition II one requires that $\Delta d(X \setminus A) \leq n$, and in condition III that $\Delta d(Z \setminus iY) \leq n$.*** This proves Theorem 1".

It was possible to dispense, in the first part of Theorem 1, with the condition of completeness of the space Y only in the case of complete subspaces A :

Theorem 1'''. For every space Y the following two conditions are equivalent:

- I. The space Y is locally connected in all dimensions $< n$ (and connected in all dimensions $< n$);
- II. If A is a complete subspace of the space X , and the difference $X \setminus A$ is precompact and $r\Delta d(X \setminus A) \leq n$, then every mapping $f : A \rightarrow Y$ can be extended to a mapping $f_U : U \rightarrow Y$ (respectively $f_X : X \rightarrow Y$), where U is some uniform neighborhood of the set A .

Remark 1. Of course, under these hypotheses there also remains true a theorem analogous to Theorem 1', on possible modifications of conditions II, and, with equal right, instead of compactness of the space X one may also use completeness.

Remark 2. For $n = 0$ every space is connected and locally connected in all dimensions $< n$. Therefore it is natural to single out the zero-dimensional case:

Theorem 0. If A is closed in the space X and $r\Delta d(X \setminus A) = 0$, then there exists a retraction $r : X \rightarrow A$; under the same hypotheses every mapping $f : A \rightarrow Y$ into an arbitrary uniform space Y has an extension $F : X \rightarrow Y$.****

Remark 3. One can construct examples showing that the condition of precompactness of the difference $X \setminus A$, as well as the condition of completeness of the space Y in Theorem 1, are essential: Theorem 1 is false if even one of them is omitted.

Remark 4. In Theorem 2, “On uniform Hannerization,” one cannot require that the inequality

$$\Delta d(Z \setminus iY) \leq \Delta d(X \setminus A)$$

hold. This can be shown by the following example. Let Y be a compact space of dimension $\Delta dY = \dim Y \geq 2$. As the space X take the interval $[0, 1]$, as the set A take the Cantor perfect set C , and as the mapping $f : A \rightarrow Y$ take a mapping of the Cantor set C onto Y , which exists by a well-known theorem of P. S. Aleksandrov. From the properties of uniform dimension it follows easily that, for any embedding $i : Y \rightarrow Z$ into a space Z and any mapping $F : X \rightarrow Z$ satisfying only condition α and the condition $F(X \setminus A) \subseteq Z \setminus iY$, the inequality

$$\Delta d(Z \setminus iY) \geq \Delta dY > \Delta d(X \setminus A) = 1$$

holds.

* For a complete space Y , the assertion of the corollary follows from certain propositions of Isbell (see ⁽⁴⁾, Ch. III, p. 17).

** The equality $\Delta d(Z \setminus iY) = \Delta d(X \setminus A)$ may fail here. See Remark 4.

*** It is unknown whether this will be true without the restriction $\Delta dY \leq n$.

**** A special case of this theorem is given in ⁽⁸⁾.

Remark 5. By virtue of Corollary 17 (⁽⁴⁾, Chapter III), the assertion of our corollary in the case of a complete space Y is valid for arbitrary uniform spaces.

Remark 6. It should be noted that all our assertions will remain true if, in each assertion, Δd is replaced by a uniform dimension δd in the sense of Smirnov ⁽⁹⁾.

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Moscow State University
named after M. V. Lomonosov

Punjabi University
Patiala, India

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