

ON BEST RATIONAL APPROXIMATIONS OF CONVEX FUNCTIONS AND FUNCTIONS OF BOUNDED VARIATION

A. P. Bulanov

1970

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-197001.40359>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 517.512

MATHEMATICS

A. P. BULANOV

ON BEST RATIONAL APPROXIMATIONS OF CONVEX FUNCTIONS AND FUNCTIONS OF BOUNDED VARIATION

(Presented by Academician A. N. Kolmogorov on 28 V 1969)

Let $R_n[f]$ and $E_n[f]$ be the least deviations (in the metric $C[a, b]$) respectively of rational functions of degree not exceeding n and polynomials of degree not exceeding n from the function $f(x)$, $x \in [a, b]$. E. P. Dolzhenko ⁽¹⁾ established that there exist continuous functions f for which $R_{n_k}[f] = E_{n_k}[f]$ for infinitely many indices n_k . At the same time f may have a modulus of continuity of any prescribed order of growth. In this connection there arises the problem of finding sufficiently simple classes of functions f for which $R_n[f]$ tends to zero essentially faster than $E_n[f]$.

We first dwell on convex functions. From a result of P. Szűsz and P. Turán ⁽²⁾ it follows: if $f(x)$ is convex on $[-1, 1]$, then on any interval $[-1 + \varepsilon, 1 - \varepsilon]$ the estimate

$$R_n[f] \leq C(\varepsilon) \ln^4 n / n^2 \quad (n = 2, 3, \dots).$$

holds.

On the other hand (see ⁽³⁾, estimate (24)), there exist such piecewise-convex functions f for which $R_n[f]$ tends to zero arbitrarily slowly. In connection with these two results the question arises: do there exist convex functions which are approximated by rational functions arbitrarily badly? The answer to this is given by

Theorem 1. For an arbitrary continuous convex function $f(x)$ ($x \in [a, b]$, $-\infty < a < b < \infty$) the estimate

$$R_n[f] \leq C_1 M \ln^2 n / n \quad (n = 2, 3, \dots), \tag{1}$$

is valid, where C_1 is an absolute constant, and M is the maximum of $|f(x)|$ on $[a, b]$.

Theorem 2. There exists a convex continuous function $f(x)$, $x \in [0, 1]$, for which the inequalities

$$R_n[f] \geq 1/n \ln^2 n \quad (n = 2, 3, \dots).$$

hold.

Theorem 2 shows that, if factors of the type $\ln^\gamma n$ ($\gamma = \text{const}$) are disregarded, estimate (1) is unimprovable.

We note in passing the estimate obtained by A. A. Gonchar ⁽⁴⁾ (see there Corollary 3 of Theorem 1):

$$R_n \left[\left(\ln \frac{e}{x} \right)^{-\gamma} \right] \leq C_2 \left(\frac{\ln n}{n} \right)^\gamma, \quad \gamma > 0, \quad x \in [0, 1].$$

It follows from Theorem 1 that the last estimate (if the factor $\ln n$ is neglected) is not sharp for $0 < \gamma < 1$ (at the same time it is unimprovable for the piecewise-convex function $f_0(x)$: $f_0(x) = 0$ for $x \in [-1, 0]$, $f_0(x) = (\ln e/x)^{-\gamma}$ for $x \in [0, 1]$).

Theorem 2a. For any sequence $\varepsilon_1 \geq \varepsilon_2 \geq \dots 0$, $\varepsilon_n \rightarrow 0$, there exists a subsequence of indices $n_1 < n_2 < \dots$ and a convex non-

continuous function $f(x)$, $x \in [0, 1]$, such that

$$R_{n_k}[f] > \varepsilon_{n_k}/n_k.$$

Theorem 3. If the convex function $f(x) \in \text{Lip } \alpha$, $\alpha > 0$, $x \in [a, b]$, then the inequalities

$$R_n[f] \leq C_3 M \frac{\ln^6 n}{n^2} \quad (n = 2, 3, \dots), \quad (2)$$

hold, where C_3 depends only on α ; M is the maximum of $|f(x)|$ on $[a, b]$.

Since D. Newman, P. Shapiro, and P. Turán [2] showed that there exists a convex function $f(x)$, $x \in [-1/2, 1/2]$, for which

$$R_n[f] > \frac{1}{n^2 \ln^2 n} \quad (n = 2, 3, \dots),$$

then (if one neglects factors of the type $\ln^\gamma n$, $\gamma = \text{const}$) estimate (2) cannot be improved. The third theorem overlaps with the following result: for continuous functions of bounded variation $f(x) \in \text{Lip } \alpha$, $x \in [a, b]$, $\alpha > 0$, the estimate* holds

$$R_n[f] \leq C_4 \frac{\ln^2 n}{n} \quad (n = 2, 3, \dots), \quad (3)$$

where C_4 does not depend on n . Both in estimate (2) and in estimate (3), on the right-hand side the Lipschitz exponent α enters only into the constants C_3 and C_4 .

Theorem 4. Let $f(x)$ ($x \in [a, b]$) be a continuous function with modulus of continuity $\omega(\delta)^{**}$, having finite total variation ($\text{Var } f < \infty$). Then for $R_n = R_n[f]$ the estimate

$$\frac{1}{|\ln R_n|} \frac{R_n}{|\ln \omega^{-1}(R_n)|} \leq \frac{C_5}{n} \quad (n \geq N(\omega)), \quad (4)$$

holds, where C_5 depends on $\text{Var } f$ and on $\omega(\delta)$; $N(\omega)$ depends on ω ; $\omega^{-1}(t)$ is the function inverse to $t = \omega(\delta)$.

Corollary. If $f(x)$, $x \in [a, b]$, is a continuous function of bounded variation whose modulus of continuity satisfies the inequality $\omega(\delta) \leq (\ln 1/\delta)^{-\gamma}$, $\gamma > 0$, then

$$R_n[f] \leq C_6 \left(\frac{\ln n}{n} \right)^{\gamma/(1+\gamma)} \quad (n = 2, 3, \dots), \quad (5)$$

where C_6 depends on $\text{Var } f$ and on γ (we note that in the work [5] of G. Freud, for functions of this class the estimate $R_n[f] \leq C_7 n^{-\gamma/(2+\gamma)}$ was obtained).

The work was carried out under the supervision of E. P. Dolzhenko.

Moscow State University
named after M. V. Lomonosov

Received
27 V 1969

CITED LITERATURE

1. E. P. Dolzhenko, *Matem. zametki*, **1**, no. 3, 403 (1962).
2. P. Szűsz, P. Turán, *MTA. Mat. Kut. Int. Közl.*, 495 (1965).
3. A. A. Gonchar, *Matem. sborn.*, **72** (114), no. 3, 489 (1967).
4. A. A. Gonchar, *Matem. sborn.*, **73** (115), no. 4, 630 (1967).
5. G. Freud, *Acta Math. Acad. Sci. Hung.*, **17** (3-4), 313 (1966).

* The result was obtained independently by A. A. Abugattarapov with E. P. Dolzhenko and G. Freud and was reported at the International Congress of Mathematicians in Moscow in 1966; see also (5).

** $\omega(\delta)$ is a nonnegative function, nondecreasing for $\delta \geq 0$, $\omega(0) = 0$. The inverse function $\omega^{-1}(t)$ is defined as follows: $\omega^{-1}(t) = \inf\{\delta : \omega(\delta) \geq t\}$.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.