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Abstract

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MATHEMATICS

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ON THE ERGODIC PRINCIPLE FOR A PARTIALLY SEMIGROUP FAMILY OF OPERATORS ON TOPOLOGICAL SEMIFIELDS

In this article we shall adhere to the definitions and notation of works ^(1,2).

1. Let a family $\{E_k, 1 \leq k \leq \infty\}$ be given, where E_k is a complete topological semifield; K_k is the cone of nonnegative elements in E_k , and ∇_k is the topological Boolean algebra of all idempotents, on which a certain measure is defined. Elements of E_k for which the integral with respect to this measure exists are called summable; denote their set by L_k . Obviously, L_k is a linear topological space in the topology induced from E_k . An element $x \in L_k \cap K_k$ will be called a distribution if $\mu(x) = 1$, where μ denotes the integral sign.

Introduce the notation

$$P_k = \{x : x \in L_k \cap K_k, \mu(x) = 1\}.$$

Consider, generally speaking, a noncommutative family $\{T_k^{[k+1]}, 1 \leq k \leq \infty\}$ of linear operators, where $T_k^{[k+1]} : L_k \rightarrow L_{k+1}$ and

$$T_k^{[m]} = T_{m-1}^{[m]} \times T_{m-2}^{[m-1]} \dots T_k^{[k+1]} \quad (k < m).$$

Definition 1. A linear operator $T_k^{[k+1]}$ ($1 \leq k \leq \infty$) will be called **stochastic** if it satisfies the following conditions:

- 1) $T_k^{[k+1]}(L_k) \subset L_{k+1}$;
- 2) $\mu(T_k^{[k+1]}x) = \mu(x)$ for $x \in L_k$;
- 3) $T_k^{[k+1]}x^{(n)} \rightarrow T_k^{[k+1]}x^{(0)}$, if $x^{(n)} \rightarrow x^{(0)}$ ($x^{(0)}, x^{(n)} \in L_k$).

Here convergence is to be understood in the sense of the topology in E_k .

Definition 2. A family $\{T_k^{[k+1]}, 1 \leq k \leq \infty\}$ of stochastic operators will be called **ergodic** if, for any k and any x, y from P_k and $g \in \nabla_2$, the relation

$$\lim_{r \rightarrow \infty} \mu(gT_k^{[r]}(x - y)) = 0$$

holds.

We shall say that a family $\{\tilde{T}_k^{[k+1]}, 1 \leq k \leq \infty\}$ of operators includes the family $\{T_k^{[k+1]}, 1 \leq k \leq \infty\}$ if, for some $i_1, i_2, \dots, i_h, \dots$ ($i_h = i_j$ when $h = j$),

$$\tilde{T}_{i_h}^{[i_{h+1}]} = T_h^{[h+1]}.$$

Definition 3. A family $\{\tilde{T}_k^{[k+1]}, 1 \leq k \leq \infty\}$ of stochastic operators will be called **strongly ergodic** if any family $\{\tilde{T}_k^{[k+1]}, 1 \leq k \leq \infty\}$ of stochastic operators that includes the family $\{T_k^{[k+1]}, 1 \leq k \leq \infty\}$ is ergodic.

Condition A. Whatever x, y from P_k may be, there exist a natural number m , elements $z \in L_m$, u and v from P_m , depending on x, y , and a number $\lambda(T_k^{[m]})$ ($0 < \lambda(T_k^{[m]}) < 1$) such that

$$T_k^{[m]}x - z = (1 - \lambda(T_k^{[m]}))u,$$

$$T_k^{[m]}y - z = (1 - \lambda(T_k^{[m]}))v,$$

where $z < T_k^{[m]}x$, $z < T_k^{[m]}y$, $k = 1, 2, \dots$

1.1. If, for the family $\{T_k^{[k+1]}, 1 \leq k \leq \infty\}$ of stochastic operators, condition (A) is satisfied, then the equality holds

$$T_1^{[n]}(x - y) = (1 - \lambda(T_1^{[m_1]}))(1 - \lambda(T_{m_1}^{[m_2]})) \dots (1 - \lambda(T_{m_{h-1}}^{[m_h]}))T_{m_h}^{[m_h+r]}(u - v),$$

$x, y \in P_1, \quad u, v \in P_{m_h} \quad \text{for } n = m_1 + m_2 + \dots + m_h + r, \quad r < m_{h+1} - m_h.$

1.2. If, for the family $\{T_k^{[k+1]}, 1 \leq k \leq \infty\}$ of stochastic operators, condition (A) is satisfied and the series

$$\lambda(T_1^{[m_1]}) + \sum_{h=2}^{\infty} \lambda(T_{m_{h-1}}^{[m_h]}) = \infty, \quad (1)$$

then for any x, y from P_1 the relation holds

$$\lim_{n \rightarrow \infty} \mu(gT_1^{[n]}(x - y)) = 0.$$

1.3. In order that the family $\{T_k^{[k+1]}, 1 \leq k \leq \infty\}$ of stochastic operators satisfying condition (A) be strongly ergodic, it is necessary and sufficient that relation (1) hold.

All known criteria of ergodicity for inhomogeneous Markov chains (see ⁽³⁾, ⁽⁴⁾, p. 206, ^(5, 6)) are sufficient conditions for strong ergodicity and constitute special cases of condition (1). From proposition 1.3, as a consequence, follow the results of papers ⁽⁷⁾ (Theorem 11), ⁽⁸⁾ (Lemma 2), ⁽⁹⁾ (Theorem 1).

2. Consider a topological semifield $E_k(t)$, depending on a numerical parameter t , and denote by $\mathcal{E}_k(t)$ the direct sum

$$\mathcal{E}_k(t) = E_k(t) \oplus iE_k(t).$$

As in item 1, it is assumed that on $\nabla_k(t)$ a certain proper measure is defined.

Let there be given, generally speaking, a noncommutative family of linear operators $\{T_k^{[k+1]}(t), 1 \leq k \leq \infty\}$, where $T_k^{[k+1]}(t) : \mathcal{L}_k(t) \rightarrow \mathcal{L}_{k+1}(t)$ and

$$T_k^{[m]}(t) = T_{m-1}^{[m]}(t) \times T_{m-2}^{[m-1]}(t) \dots T_k^{[k+1]}(t)$$

for $k < m$. Here $\mathcal{L}_k(t)$ is the set of summable elements of the complex semifield $\mathcal{E}_k(t)$.

Definition 4. A family $\{T_k^{[k+1]}(t), 1 \leq k \leq \infty\}$ of linear operators continuously depending on the numerical parameter t will be called a **family of characteristic operators** if the following relations are fulfilled:

- 1) $T_k^{[k+1]}(t)(\mathcal{L}_k(t)) \subseteq \mathcal{L}_{k+1}(t)$;
- 2) $T_k^{[k+1]}(0) = T_k^{[k+1]}$ is a stochastic operator;
- 3) $T_k^{[k+1]}(t)x^{(n)}(t) \rightarrow T_k^{[k+1]}(t)x^{(0)}(t)$, if

$$x^{(n)}(t) \rightarrow x^{(0)}(t) \quad (x^{(0)}(t), x^{(n)}(t) \in \mathcal{L}_k(t)).$$

If, for the family $\{T_k^{[k+1]}(t), 1 \leq k \leq \infty\}$ of characteristic operators, a condition analogous to (A), taking into account the parameter t , is satisfied, then results 1.1 and 1.2 remain valid for this family.

From the results of item 2, as a consequence, follows the result of paper ⁽¹⁰⁾ (Lemma 5).

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