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Abstract

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MATHEMATICS

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BOUNDARY-VALUE PROBLEMS FOR ELIPTIC AND PARABOLIC EQUATIONS IN UNBOUNDED DOMAINS

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I. Let G be the exterior of a bounded domain in R^n . The boundary of G is a smooth $(n - 1)$ -dimensional surface Γ . We shall consider the boundary-value problem

$$A(x, D)u = \sum_{|\alpha| \leq 2l} a_\alpha(x) D^\alpha u = f(x), \quad x \in \bar{G}; \quad (1)$$

$$B_j(x, D)u = \sum_{|\beta| \leq m_j} b_{j\beta}(x) D^\beta u = g_j(x), \quad x \in \Gamma, \quad (2)$$

where

$$j = 1, \dots, l; \quad \alpha = (\alpha_1, \dots, \alpha_k), \quad |\alpha| = \sum_{k=1}^n \alpha_k,$$

$$D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}, \quad D_k^{\alpha_k} = \frac{1}{(-i)^{\alpha_k}} \frac{\partial^{\alpha_k}}{\partial x_k^{\alpha_k}}.$$

We assume that the following conditions are satisfied:

1. The operator A is uniformly elliptic in $\bar{G}$.
2. At every point $x \in \Gamma$ the Shapiro-Lopatinskii condition is satisfied.
3. The limit $\lim_{|x| \rightarrow \infty} a_\alpha(x)$ exists for all α , $|\alpha| \leq 2l$.
4. The symbol of the limiting operator

$$a(\infty, \xi) = \sum_{|\alpha| \leq 2l} a_\alpha(\infty) \xi^\alpha$$

has no zeros for all $\xi \in R^n$.

5. The coefficients $a_\alpha(x)$ and $b_{j\beta}(x)$ are infinitely smooth functions. The spaces $H^k(G)$ and $H^{k-1/2}(\Gamma)$ are the usual Sobolev-Slobodetskii spaces.

Theorem 1. Suppose that conditions 1-5 are satisfied. Then:

- a) for any $u \in H^k(G)$ the inequality

$$\|u\|_k(G) \leq C \left\{ \|f\|_{k-2l}(G) + \sum_{j=1}^l \|g_j\|_{k-m_j-1/2}(\Gamma) + \|u\|_{k-1}(G_0) \right\},$$

holds, where $k \geq \max(2l, m_j + 1/2)$, $G_0 = G \cap K_R$; $K_R = \{x : |x| \leq R\}$;

- b) there exists a regularizer for problem (1), (2), i.e., a bounded operator

$$R : H^{k-2l}(G) \times H^{k-m_j-1/2}(\Gamma) \rightarrow H^k(G)$$

such that $LR\Phi = \Phi + T\Phi$, where $L = (A, B_j)$, $\Phi = (f, g_j)$, and T is a completely continuous operator in the direct product

$$H^{k-2l}(G) \times H^{k-m_j-1/2}(\Gamma).$$

The space $H_a^k(G)$ is the space of functions with norm

$$\|u\|_{k,a}(G) = \sum_{|\alpha| \leq k} \iint |D^\alpha u|^2 e^{2a|x|} dx.$$

Theorem 2. Suppose that conditions 1-5 are satisfied and that the polynomial $a(\infty, \xi + i\tau)$ has no real zeros in ξ for $|\tau| \leq a$. Then:

- a) for any function $u(x) \in H_a^k(G)$ the estimate is valid

$$\|u\|_{k,a}(G) \leq C \left\{ \|f\|_{k-2l,a}(G) + \sum_1^l \|g_j\|_{k-m_j-1/2}(\Gamma) + \|u\|_{k-1}(G_0) \right\};$$

- b) there exists a regularizer of the problem (1), (2), acting from the space $H_a^{k-2l}(G) \times H^{k-m_j-1/2}$ into the space $H_a^k(G)$.

II. In the same domain with boundary Γ we consider the elliptic problem with complex parameter

$$A(x, D, q)u(x, q) = \sum_{|\alpha|+\beta \leq 2l} a_{\alpha\beta}(x) q^\beta D^\alpha u = f(x, q), \quad x \in \bar{G}; \quad (3)$$

$$B_j(x, D, q)u(x, q) = \sum_{|\beta|+|\gamma| \leq m_j} q^\gamma b_{j\beta\gamma}(x) D^\beta u = g_j(x, q), \quad x' \in \Gamma. \quad (4)$$

The analogous problem in a bounded domain was considered in the work ⁽¹⁾. The parameter q varies in the angle $Q : \varphi_1 \leq \arg q \leq \varphi_2$. We assume that the following conditions are satisfied:

6. The coefficients $a_{\alpha\beta}(x)$ and $b_{j\beta\gamma}(x)$ are infinitely differentiable functions in $\bar{G}$, and the inequalities $|D^\gamma a_{\alpha\beta}| \leq C_{a,\gamma}^\beta$ hold, where $C_{a,\gamma}^\beta$ are constants.
7. The operator A is elliptic with parameter, and the boundary-value problem A, B_j satisfies the Shapiro-Lopatinskii condition with parameter (for the exact definition see Chap. I, § 4 of ⁽¹⁾).

Theorem 3. *Let conditions 6 and 7 be satisfied and let k be an integer $k \geq k_0 = \max(2l, m_j + 1)$. Then there exists a number $q_1 > 0$ such that for $|q| > q_1$, $q \in Q$, problem (3), (4) has a unique solution $u(x) \in H^k(G)$, if $f(x) \in H^{k-2l}(G)$, and $g_j(x) \in H^{k-m_j-1/2}(\Gamma)$. Moreover, the inequality holds*

$$\|u\|_k(G) + |q|^k \|u\|_0(G) \leq C \left\{ \|f\|_{k-2l} + |q|^{k-2l} \|f\|_0 + \sum_1^l \left(\|g_j\|_{k-m_j-1/2}(\Gamma) + |q|^{k-m_j-1/2} \|g_j\|_0(\Gamma) \right) \right\}.$$

The constant C does not depend on $u(x)$ and q .

Remark. An analogous result is valid for the case of an unbounded domain G with unbounded boundary Γ .

III. In the author's work ⁽³⁾ problems for elliptic equations with unbounded coefficients in the exterior of a bounded domain were considered. Here we shall study the analogous problem with parameter. To this end, assume that problem (3), (4) satisfies the following conditions:

8. The inequality holds

$$\left| \sum_{|\alpha|+\beta=2l} a_{\alpha\beta}(x) \xi^\alpha q^\beta \right| \geq \delta(x) (|\xi|^2 + |q|^2)^l$$

for any $\xi \in R^n$, $q \in Q = \{q : \varphi_1 \leq \arg q \leq \varphi_2\}$, $\delta(x) \geq \delta_0 > 0$.

9. The function

$$a_{2l}(x, \xi, q) = \sum_{|\alpha|+\beta=2l} a_{\alpha\beta}(x) q^\beta \xi^\alpha$$

for $|x| > N$, $q \in Q$, $\xi \in R^n$ leaves free of values the angle of the complex plane $Q_1 = \{z : \psi_1 \leq \arg z \leq \psi_2\}$.

10. The values of the function $a_{0,0}(x)$ for $|x| > N$ lie in the angle $Q_2 = \{z : \psi_1 + \pi + \varepsilon \leq \arg z \leq \psi_2 + \pi - \varepsilon\}$, $\varepsilon > 0$.

11. At every point $x' \in \Gamma$ the Shapiro-Lopatinskii conditions with parameter are satisfied.
12. Let x be any point of G , $|x| > N$; H a cube with edge h centered at the point x ; y any point of H . We require that the functions $|a_{\alpha\beta}(x) - a_{\alpha\beta}(y)|/\delta(x)$ for $|\alpha| + \beta = 2l$, $|a_{0,0}(x) - a_{0,0}(y)|/|a_{0,0}(x)|$ tend to zero as $h \rightarrow 0$ uniformly in x , $|x| \geq N$, $x \in G$.

Introduce the notation $\delta_k(x) = (\delta(x))^{k/2l}$, $\gamma_k(x) = (|a_{0,0}(x)|)^{k/2l}$.

13. Let $|\alpha| + \beta = 2l - k$, and $|\gamma| = p$; then the ratio $|D^\gamma a_{\alpha\beta}(x)|/\delta_{2l-(k+p)}\gamma_{k+p} \rightarrow 0$ as $|x| \rightarrow \infty$, if $0 \leq p \leq s - 2l$ for $0 < k < 2l$ and $0 < p \leq s - 2l$ for $k = 0, 2l$.

Conditions 12 and 13 allow certain classes of unbounded functions. Condition 12 is satisfied by polynomials and functions of the type $e^{k|x|}$.

For convenience we shall use special norms depending on the parameter q . For each fixed q they are equivalent to the norms $\|u\|_k(G)$, which were used by the author. The definition of the norm is as follows:

$$\langle u \rangle_k^2(G) = \iint \{ \delta_k^2(x) |D^k u|^2 + |q|^{2k} |u|^2 + \gamma_k^2(x) |u|^2 \} dx.$$

Theorem 4. Suppose conditions 8–13 are fulfilled. Then there exists a number $q_1 > 0$ such that, for $|q| > q_1$, $q \in Q$, there exists a unique solution of problem (3), (4) belonging to the space $H^k(\delta, \gamma, G)$, if $f \in H^{k-2l}(\delta, \gamma, G)$, and $g_j \in H^{k-m_j-1/2}(\Gamma)$. Moreover, the estimate

$$\langle u \rangle_k(G) \leq C \left\{ \langle f \rangle_{k-2l}(G) + \sum_1^l \|g_j\|_{k-m_j-1/2}(\Gamma) \right\}.$$

holds.

The constant C does not depend on $u(x)$ and q .

An analogous result is also true in the case of an unbounded boundary.

- IV. After studying the elliptic problem with a parameter, it is natural to investigate the general mixed problem for a parabolic equation in a cylinder with unbounded base. Let G , as before, be the exterior of a bounded domain with boundary Γ , and let Ω_T be the cylinder $\Omega_T = G \times (0, T)$ with lateral surface $\Omega'_T = \Gamma \times (0, T)$.

Consider the parabolic problem

$$A(x, t, D_x, \partial/\partial t)u(x, t) = f(x, t), \quad (x, t) \in \Omega_T; \quad (5)$$

$$B_j(x, t, D_x, \partial/\partial t)u(x, t)|_{x=x'} = g_j(x', t), \quad (x', t) \in \Omega'_T; \quad j = 1, \dots, l. \quad (6)$$

For $t = 0$ we prescribe the initial conditions

$$\partial^k u / \partial t^k|_{t=0} = \varphi_k(x) \quad (k = 0, \dots, \chi - 1, \quad x \in G), \quad \chi = l/b. \quad (7)$$

The operator A has order $2l$, and the operators B_j have order m_j . The parabolic weight of the problem is $2b$.

Definition. Problem (5), (6) is called **normally parabolic** if, for each fixed $t = a$, $0 \leq a \leq T$, the problem with a parameter in the domain G , obtained from problem (5), (6) by replacing $\partial/\partial t$ with q^{2b} , satisfies all conditions 8–13 in the angle $|\arg q| \leq \pi/4b$.

We shall use on the surface Ω'_T the usual Sobolev–Slobodetskii space $H_{\lambda, \lambda/2b}(\Omega'_T)$, and in the cylinder Ω_T the space $H_{s, s/2b}(\delta, \gamma, \Omega_T)$ with norm

$$\langle u \rangle_{s, s/2b}^2 = \int_{\Omega_T} \{ \delta_s^2(x) (|D_x^s u|^2 + |D_t^{s/2b} u|^2) + \gamma_s^2(x) |u|^2 \} dx dt.$$

It is assumed that $s, s/2b$ are integers.

The discussion is carried out by the method of the Laplace transform, analogously to how this was done in paper ⁽¹⁾.

Theorem 5. *Let problem (5), (6), (7) be a normally parabolic problem. Suppose also that the usual compatibility conditions for the right-hand sides are fulfilled. Then, if $f(t, t) \in H_{\mu, \mu/2b}(\delta, \gamma, \Omega_T)$; $g_j(x', t) \in H_{\lambda_j, \lambda_j/2b}(\Omega'_T)$;*

$\varphi_k(x) \in H_{s-2bk-b}(G)$, $k = 0, \dots, \kappa - 1$, where $\mu = s - 2l$, $\lambda_j = s - m_j - \frac{1}{2}$, then there exists a unique solution of the problem $u(x, t) \in H_{s, s/2b}(\delta, \gamma, \Omega_T)$. Moreover, the a priori estimate holds

$$\langle u \rangle_{s, s/2b} \leq C \left\{ \langle f \rangle_{\mu, \mu/2b} + \sum_1^l \|g_j\|_{\lambda_j, \lambda_j/2b} + \sum_1^{\kappa-1} \|\varphi_k\|_{s-2bk-b} \right\}.$$

The constant C does not depend on $u(x, t)$.

V. Now in the same domain G with boundary Γ consider problem (1), (2). We shall assume that conditions 1, 2, and 5 are satisfied. Conditions 3 and 4 will be replaced by the following.

Condition 9°. The coefficients of equation (1) and their derivatives have the form

$$D^\beta a_\alpha(x) = a_{\alpha\beta}^0(\omega) r^{|\alpha| - |\beta| - 2l} + a_{\alpha\beta}^1(x); \quad r = |x|.$$

Here $\omega = (\omega_1, \dots, \omega_{n-1})$ is an infinitely differentiable coordinate system on the unit sphere, and

$$a_{\alpha\beta}^1(x) = O(r^{|\alpha| - |\beta| - 2l - \delta})$$

as $r \rightarrow \infty$; $\delta > 0$.

Consider in R^n the elliptic equation $\sum a_\alpha^0(\omega) D^\alpha u = f$. Make the change of variables $x \rightarrow (r, \omega)$, and then the change $r = e^{-t}$. It is well known that after these changes we obtain the equation

$$A^0(\omega, D_\omega, D_t)v(t, \omega) = e^{-2lt}f(t, \omega) \quad (8)$$

in the cylinder $\Omega = S \times (-\infty, +\infty)$, where S is the unit sphere $|x| = 1$. After the Fourier transform in t we obtain an elliptic equation with a parameter on the sphere:

$$A^0(\omega, D_\omega, \lambda)\tilde{v}(\lambda, \omega) = \tilde{f}_1(\lambda, \omega). \quad (9)$$

It is known that the inverse operator $R(\lambda)$ for equation (9) is a meromorphic function of λ , and in each strip $|\operatorname{Im} \lambda| < C_1$ there exists only a finite number of poles. The spaces $\dot{W}_\alpha^k(G)$ are the same as those used by V. A. Kondrat'ev in paper (2).

Theorem 6. Suppose conditions 1, 2, 5, and 9 are satisfied, and suppose that the function $R(\lambda)$ of problem (9) has no poles on the line

$$\operatorname{Im} \lambda = \frac{1}{2}(-\alpha - n + 2k), \quad k \geq k_0 = \max(2l, m_j + \frac{1}{2}).$$

Then, if

$$(f, g_j) \in \dot{H}_\alpha^k = \dot{W}_\alpha^{k-2l} \times H^{k-m_j-1/2},$$

then, provided a finite number of conditions of the form $F_i(f, g_j) = 0$ are satisfied, where F_i are functionals in $\dot{H}_\alpha^k(G)$, there exists a solution of problem (1), (2) belonging to $\dot{W}_\alpha^k(G)$.

Under the same assumptions, there hold a theorem on the increase of smoothness, a theorem on the finite-dimensionality of the space of solutions of the homogeneous problem, and a theorem on the asymptotic expansion of the solution as $r \rightarrow \infty$. The methods of proof are analogous to the methods of proof in paper (2).

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