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Abstract

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MATHEMATICS

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THE CONJUGATION PROBLEM FOR THREE ANALYTIC FUNCTIONS

(Presented by Academician N. I. Muskhelishvili on May 4, 1970)

Let L be a simple smooth closed contour dividing the complex plane into an interior domain S^+ , containing the point $z = 0$, and an exterior domain S^- , containing the point at infinity. Let us construct one more exterior domain E^- as follows. Join the points $z = 0$ and $z = \infty$ by a simple cut intersecting L at the single point t_0 . In what follows, by $\sqrt{z}$ we shall mean the fixed branch having a discontinuity along the indicated cut. The locus of the points $\sqrt{t}$ and $(-\sqrt{t})$, where t runs over L , forms a simple smooth closed contour Γ . As E^- we take the domain lying outside this contour. It is obvious that if $z \in E^-$, then also $(-z) \in E^-$.

Problem. Find functions $\Phi^+(z)$ and $\Psi^+(z)$, analytic in S^+ and continuous in $S^+ + L$, and also a function $\Phi^-(z)$, analytic in E^- , continuous in $E^- + \Gamma$, and equal to zero at infinity, under the boundary conditions

$$\begin{aligned}\Phi^+(t) &= A(t)\Phi^-(\sqrt{t}) + G(t), \\ \Psi^+(t) &= B(t)\Phi^-(\sqrt{t}) + H(t),\end{aligned}\quad t \in L, \quad (1)$$

where $A(t)$, $B(t)$, $G(t)$, and $H(t)$ are functions given on L satisfying a Hölder condition; $A(t)$ and $B(t)$ do not vanish anywhere on L .

Below it is shown that problem (1), in its nature and in the method of solution, is related to the well-known conjugation problem (the Riemann problem) ^(1,2).

1. Let us first consider the "jump" problem

$$\begin{aligned}\varphi^+(t) &= \varphi^-(\sqrt{t}) + g(t), & \psi^+(t) &= \varphi^-(\sqrt{t}) + h(t), \\ t &\in L, & \varphi^-(\infty) &= 0.\end{aligned}\quad (2)$$

Introduce the new unknown functions

$$\varphi_1^-(z) = \varphi^-(\sqrt{z}) - \varphi^-(-\sqrt{z}), \quad \varphi_2^-(z) = 1/\sqrt{z} [\varphi^-(\sqrt{z}) - \psi^-(-\sqrt{z})],$$

$$z \in S^-.$$

These functions are analytic in S^- , continuous in $S^- + L$, and equal to zero at infinity. Setting further $\varphi_1^+ = \varphi^+ + \psi^+$, $\varphi_2^+ = \varphi^+ - \psi^+$, from conditions (2) we arrive at two independent conjugation problems

$$\varphi_1^+(t) = \varphi_1^-(t) + g(t) + h(t), \quad t \in L; \quad (3)$$

$$\varphi_2^+(t) = \sqrt{t} \varphi_2^-(t) + g(t) - h(t), \quad t \in L. \quad (4)$$

Solving problems (3) and (4) by the known method and using the results of N. I. Muskhelishvili on the behavior of a Cauchy-type integral with density having a discontinuity (see, in particular, ⁽¹⁾, p. 77), we arrive at the theorem.

Theorem 1. Let the given function $g(t) + h(t)$ satisfy the Hölder condition everywhere on L , and let the given function $g(t) - h(t)$ satisfy

satisfies this condition on L everywhere except, possibly, the point t_0 , where a discontinuity of the first kind is allowed for this function.

Then problem (2) has, in the class of functions analytic in the above-indicated domains and continuous up to the boundary, a unique solution (L is traversed counterclockwise):

$$\varphi^+(z) = \frac{1}{4\pi i} \int_L \frac{g(\tau) + h(\tau)}{\tau - z} d\tau + \frac{\sqrt{z - t_0}}{4\pi i} \int_L \frac{g(\tau) - h(\tau)}{\sqrt{\tau - t_0}(\tau - z)} d\tau, \quad z \in S^+; \quad (5)$$

$$\psi^+(z) = -\frac{1}{4\pi i} \int_L \frac{g(\tau) + h(\tau)}{\tau - z} d\tau - \frac{\sqrt{z - t_0}}{4\pi i} \int_L \frac{g(\tau) - h(\tau)}{\sqrt{\tau - t_0}(\tau - z)} d\tau, \quad z \in S^+; \quad (6)$$

$$\varphi^-(z) = \frac{1}{4\pi i} \int_L \frac{g(\tau) + h(\tau)}{\tau - z^2} d\tau + \frac{z\sqrt{z^2 - t_0}}{4\pi i\sqrt{z^2}} \int_L \frac{g(\tau) - h(\tau)}{\sqrt{\tau - t_0}(\tau - z^2)} d\tau, \quad z \in E^-. \quad (7)$$

The limiting values $\varphi^+(t)$, $\psi^+(t)$, and $\varphi^-(t)$ satisfy the Hölder condition. $\sqrt{w - t_0}$ denotes the branch of the root having a discontinuity along a part of the cut indicated earlier, connecting t_0 with infinity.

2. We proceed to the problem of factorization of the coefficients $A(t)$ and $B(t)$. By analogy with (5)–(7), we write formulas in which the logarithms are single-valued functions, having, possibly, discontinuities at the single point t_0 :

$$X_0^+(z) = \exp \left\{ \frac{1}{4\pi i} \int_L \frac{\ln A(\tau)B(\tau)}{\tau - z} d\tau + \frac{\sqrt{z - t_0}}{4\pi i} \int_L \ln \frac{A(\tau)}{B(\tau)} d\tau / \sqrt{\tau - t_0} \times (\tau - z) \right\}, \quad z \in S^+; \quad (8)$$

$$\Omega_0^+(z) = \exp \left\{ \frac{1}{4\pi i} \int_L \frac{\ln A(\tau)B(\tau)}{\tau - z} d\tau - \frac{\sqrt{z - t_0}}{4\pi i} \int_L \ln \frac{A(\tau)}{B(\tau)} d\tau / \sqrt{\tau - t_0} \times (\tau - z) \right\}, \quad z \in S^+; \quad (9)$$

$$X_0^-(z) = \exp \left\{ \frac{1}{4\pi i} \int_L \frac{\ln A(\tau)B(\tau)}{\tau - z} d\tau + \frac{z\sqrt{z^2 - t_0}}{4\pi i\sqrt{z^2}} \int_L \ln \frac{A(\tau)}{B(\tau)} d\tau / \sqrt{\tau - t_0} \times (\tau - z^2) \right\}, \quad z \in E^-. \quad (10)$$

The properties of the functions X_0^+ , Ω_0^+ , and X_0^- depend on the index

$$\varkappa = \text{Ind } A(t) + \text{Ind } B(t) = \frac{1}{2\pi i} [\ln A(\tau)B(\tau)]_L. \quad (11)$$

Let the function $(z - t_0)^{\varkappa/2}$ be analytic in S^+ , and let the function $\omega(z) = (z^2 - t_0)^{\varkappa/2}$ be analytic in E^- , with the possible exception of the point $z = \infty$, and satisfy the condition $\omega(1/\bar{t}) = (t - t_0)^{\varkappa/2}$ for $t \in L$. On the basis of formulas of N. I. Muskhelishvili ((1), § 22) we obtain

Theorem 2. The functions

$$X^+(z) = (z - t_0)^{\varkappa/2} X_0^+(z), \quad \Omega^+(z) = (-1)^{\varkappa} (z - t_0)^{-\varkappa/2} \Omega_0^+(z) \quad (12)$$

are analytic in S^+ and continuous in $S^+ + L$; the limiting values $X^+(t)$ and $\Omega^+(t)$ satisfy the Hölder condition on L .

The function

$$X^-(z) = (z^2 - t_0)^{-\varkappa/2} X_0^-(z) \quad (13)$$

is analytic in E^- (for $\varkappa < 0$ it has a pole of order $|\varkappa|$ at infinity), and is continuously extendable to the contour Γ ; the limiting value $X^-(t)$ satisfies the Hölder condition on Γ .

The formulas are valid

$$X^+(t) = A(t)X^-(\sqrt{t}), \quad \Omega^+(t) = B(t)X^-(\sqrt{t}), \quad t \in L. \quad (14)$$

The functions (12) and (13) play the role of the canonical solution of the conjugation problem. These functions do not vanish anywhere at a finite distance.

3. Let us proceed to the direct solution of problem (1). Using the factorization (14), we represent condition (1) in the form

$$\Phi^+(t)/X^+(t) = \Phi^-(\sqrt{t})/X^-(\sqrt{t}) + G(t)/X^+(t),$$

$$\Psi^+(t)/\Omega^+(t) = \Phi^-(\sqrt{t})/X^-(\sqrt{t}) + H(t)/\Omega^+(t), \quad t \in L. \quad (15)$$

Consider the unknown function $\Phi^-(z)/X^-(z)$. On the basis of the properties of the function $X^-(z)$ and of the condition $\Phi^-(\infty) = 0$ adopted by us, we conclude that the function $\Phi^-(z)/X^-(z)$ for $\varkappa \leq 0$ has at infinity a zero of order not less than $|\varkappa| + 1$, while for $\varkappa > 0$ this function may have a pole of order not greater than $\varkappa - 1$. Therefore the indicated function can be represented in the form

$$\Phi^-(z)/X^-(z) = \varphi^-(z) + P_{\varkappa-1}(z), \quad (16)$$

where $\varphi^-(z)$ vanishes at infinity, and $P_{\varkappa-1}(z)$ is a polynomial of degree not exceeding $\varkappa - 1$. For $\varkappa \leq 0$ we put $P_{\varkappa-1} \equiv 0$.

Introducing also the notation

$$\Phi^+(z)/X^+(z) = \varphi^+(z), \quad \Psi^+(z)/\Omega^+(z) = \psi^+(z), \quad (17)$$

we give conditions (15) the form of the jump problem (2), where now

$$g(t) = G(t)/X^+(t) + P_{\varkappa-1}(\sqrt{t}), \quad h(t) = H(t)/\Omega^+(t) + P_{\varkappa-1}(-\sqrt{t}). \quad (18)$$

Using Theorem 1, we arrive at the following results.

Theorem 3. *If the index $\varkappa$ defined by formula (11) is positive, then problem (1) is unconditionally solvable, and the homogeneous problem has exactly $\varkappa$ linearly independent solutions. The general solution can be constructed in quadratures by formulas (16), (17), (5)–(7), (18), (12), (13), and (8)–(10), where $P_{\varkappa-1}(z)$ is a polynomial of degree $\varkappa - 1$ with arbitrary complex coefficients.*

Theorem 4. *In the case $\varkappa = 0$, problem (1) is unconditionally solvable and has a unique solution. This solution can be constructed in quadratures by the formulas indicated in Theorem 3, where $P_{\varkappa-1} \equiv 0$.*

As an example, let us write out the function $\Phi^-(z)$. Using equality (16), where $P_{\varkappa-1} \equiv 0$, we obtain

$$\Phi^-(z) = X^-(z) \left\{ \frac{1}{4\pi i} \int_L \left(\frac{G(\tau)}{X^+(\tau)} + \frac{H(\tau)}{\Omega^+(\tau)} \right) \frac{d\tau}{\tau - z^2} + \right. \\ \left. + \frac{z\sqrt{\bar{z}^2 - t_0}}{4\pi i\sqrt{\bar{z}^2}} \int_L \left(\frac{G(\tau)}{X^+(\tau)} - \frac{H(\tau)}{\Omega^+(\tau)} \right) \frac{d\tau}{\sqrt{\tau - t_0}(\tau - z^2)} \right\}, \quad z \in E^-. \quad (19)$$

We shall also arrive at an expression of the form (19) when solving problem (1) in the case $\varkappa < 0$. However, in the latter case the function $X^-(z)$ has at infinity a pole of order $|\varkappa|$, as a result of which problem (1) will be solvable not for all functions $G(t)$ and $H(t)$ satisfying the Hölder condition. In the usual way (cf. (2), pp. 118–119) we derive the necessary and sufficient solvability conditions:

$$\int_L (G(\tau)/X^+(\tau) + H(\tau)/\Omega^+(\tau)) \tau^{k-1} d\tau = 0, \quad k = 1, 2, \dots, -[(\varkappa + 1)/2]. \quad (20)$$

$$\int_L \left(\frac{G(\tau)}{X^+(\tau)} - \frac{H(\tau)}{\Omega^+(\tau)} \right) \left\{ \frac{d^{k-1}}{dz^{k-1}} \frac{\sqrt{1 - t_0 z}}{1 - \tau z} \right\}_{z=0} \frac{d\tau}{\sqrt{\tau - t_0}} = 0,$$

$$k = 1, 2, \dots, [(-\varkappa + 1)/2]. \quad (21)$$

Here the symbol $[x]$ denotes the greatest integer not exceeding x .

Theorem 5. *In the case $\varkappa < 0$, for the solvability of problem (1) the conditions (20) and (21), whose number is equal to $|\varkappa|$, are necessary and sufficient. If these conditions are fulfilled, then the problem has a unique solution, which can be obtained from the formulas indicated in Theorem 3, where $P_{\varkappa-1} \equiv 0$.*

Let us note that the limiting values $\Phi^+(t)$, $\Psi^+(t)$, and $\Phi^-(t)$ of the functions obtained as a result of solving problem (1) satisfy the Hölder condition and, moreover, $\Phi^-(\gamma t_0) = \Phi^-(-\gamma t_0)$.

4. The problem considered is a particular case of the following problem of conjugation of analytic functions, whose number is equal to $n + 1$. Suppose that on different copies of the complex plane there are given $(n + 1)$ domains $S_1^+, \dots, S_n^+$ and E^- , bounded by simple smooth closed contours $L_1, \dots, L_n$ and Γ . Any of the indicated domains may contain the infinitely distant point. On the contours a positive direction of traversal is specified so that the corresponding domains remain on the left. We divide Γ into

the sum of n simple open arcs γ_k , $k = 1, \dots, n$, and on the contours L_k we specify one point t_k , $k = 1, \dots, n$. Let $\alpha_k(t)$ be functions defined on L_k , one-to-one (with the exception of t_k) and continuously mapping L_k onto γ_k with a change of the direction of traversal, $k = 1, \dots, n$. The points t_k correspond to the endpoints of the arcs γ_k . It is required to find functions $\Phi_1^+(z), \dots, \Phi_n^+(z)$, analytic in the domains $S_1^+, \dots, S_n^+$, respectively, continuous in $S_k^+ + L_k$, and also a function $\Phi^-(z)$, analytic in E^- , continuous in $E^- + \Gamma$, from n boundary conditions

$$\Phi_k^+(t) = A_k(t)\Phi^-[\alpha_k(t)] + G_k(t), \quad t \in L_k, \quad k = 1, \dots, n. \quad (22)$$

Here $A_k(t)$ and $G_k(t)$ are given functions on L_k , satisfying the Hölder condition, $k = 1, \dots, n$. It is also assumed that there exist derivatives $\alpha'_k(t)$ satisfying the Hölder condition and having no zeros.

In the case when all the domains S_k^+ coincide, and the functions $\alpha_k(t)$ are formed from different branches of the function $z^{1/n}$, problem (22), like problem (1), admits a solution in quadratures.

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CITED LITERATURE

1. N. I. Muskhelishvili, *Singular Integral Equations*, Moscow, 1968.
2. F. D. Gakhov, *Boundary Value Problems*, Moscow, 1963.

Note: Figure translations are in progress. See original paper for figures.

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