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Abstract

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MATHEMATICS

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ON EMBEDDING THEOREMS FOR BESOV SPACES WITH MIXED NORM

(Presented by Academician S. L. Sobolev on February 17, 1970)

Embedding theorems for the spaces $W_{(p)}^l$ (l integer), $H_{(p)}^l$, $S_{(p)}^r H$, $L_{(p),(\theta)}^r$ with mixed norm have been considered in a number of works ⁽⁵⁻¹⁵⁾. In the present note embedding theorems are given for the spaces $B_{(p)}^l(E^n)$ with mixed norm.

Weighted spaces $B_{(p),\alpha}^l(E^n)$ are also considered. Let E^n be the n -dimensional real Euclidean space of points $\mathbf{x} = (x_1, \dots, x_n)$; $E^+ = \{\mathbf{x} : x_n > 0\}$; $E^n = E^{n_1} \times \dots \times E^{n_k} = E^{m_1} \times \dots \times E^{m_\tau}$; $E^+ = \tilde{E}^{n_1} \times \dots \times \tilde{E}^{n_k} = \tilde{E}^{m_1} \times \dots \times \tilde{E}^{m_\tau}$, and the decompositions into E^{n_i} and E^{m_j} ($\tilde{E}^{n_i}$ and $\tilde{E}^{m_j}$) ($i = 1, \dots, k$; $j = 1, \dots, \tau$) do not depend on one another; the vectors of the subspaces E^{n_i} and E^{m_j} will be denoted respectively by $\mathbf{x}_{n_i} = (x_{i1}, \dots, x_{in_i})$ and $\mathbf{x}_{m_j} = (x_1^{(j)}, \dots, x_{m_j}^{(j)})$ ($i = 1, \dots, k$; $j = 1, \dots, \tau$); $\mathbf{p} = (p_1, \dots, p_k)$, $\mathbf{q} = (q_1, \dots, q_\tau)$, $\mathbf{l} = (l_1, \dots, l_n)$, $\mathbf{r} = (r_1, \dots, r_n)$; $\nu_i, \bar{l}_i, \bar{r}_i$ ($i = 1, \dots, n$) are nonnegative integers, $\alpha \geq 0$.

$\square_h^{n_i}(\mathbf{x}_{n_i})$ is an n_i -dimensional parallelepiped in E^{n_i} with vertex at the point $\mathbf{x}_{n_i}$ and edges $h^{n_{i1}}, \dots, h^{n_{in_i}}$ ($i = 1, \dots, k$).

Let $f(y)$ be a smooth finite function in E^n and

$$\|f\|_{\mathcal{L}_{(p),\alpha}^{l_i}(E^n)} = \left(\int_0^\infty \frac{dt}{t^{1+p_k\alpha_i}} \left\| y_n^{\alpha/p_1} \Delta_i^2(t) D_i^{\bar{l}_i} f(y) \right\|_{L_{(p)}(E^n)}^{p_k} \right)^{1/p_k} < \infty \quad (1)$$

(where $L_{(p)}(E^n)$ is the space with mixed norm (see (8)));

$$\|f\|_{B_{(p),\alpha}^l(E^n)} = \|f\|_{L_{(p)}(E^n)} + \sum_{i=1}^n \|f\|_{\mathcal{L}_{(p),\alpha}^{l_i}(E^n)}. \quad (2)$$

Putting in (2) $\alpha = 0$ and replacing E^+ by E^n , we obtain

$$\|f\|_{B_{(p)}^l(E^n)} = \|f\|_{L_{(p)}(E^n)} + \sum_{i=1}^n \|f\|_{\mathcal{L}_{(p)}^{l_i}(E^n)}. \quad (3)$$

By the spaces $B_{(p)}^l(E^n)$ and $B_{(p),\alpha}^+(E^n)$ we shall mean the closures of the set of smooth finite functions in the norms (2) and (3).

We formulate the main results.

Theorem 1. If $1 < p_i \leq q_i < \infty$ ($i = 1, \dots, k$; $j = 1, \dots, \tau$); $l_i = \bar{l}_i + \alpha_i$, $0 < \alpha_i \leq 1$ ($i = 1, \dots, n$);

$$1 - \sum_{i=1}^n \frac{1}{l_i} (1 + \nu_i) + \sum_{i=1}^k \frac{1}{p_i} \sum_{j=1}^{n_i} \frac{1}{l_{ij}} + \sum_{i=1}^{\tau} \frac{1}{q_i} \sum_{j=1}^{m_i} \frac{1}{l_j^{(i)}} > 0,$$

$f \in B_{(p)}^l(E^n)$, then $D_1^{\nu_1} \dots D_n^{\nu_n} f \in L_{(q)}(E^n)$, and the inequality

$$\|D_1^{\nu_1} \dots D_n^{\nu_n} f\|_{L_{(q)}(E^n)} \leq C \|f\|_{B_{(p)}^l(E^n)} \quad (4)$$

holds.

Theorem 2. If $1 < p_i \leq q_j < \infty$ ($i = 1, \dots, k$; $j = 1, \dots, \tau$); $l_i = \bar{l}_i + \alpha_i$, $0 < \alpha_i \leq 1$, $r_i = \bar{r}_i + \beta_i$, $0 < \beta_i \leq 1$ ($i = 1, \dots, n$);

$$1 - \sum_{i=1}^n \frac{1}{l_i} + \sum_{i=1}^k \frac{1}{p_i} \sum_{j=1}^{n_i} \frac{1}{l_{ij}} + \sum_{i=1}^{\tau} \frac{1}{q_i} \sum_{j=1}^{m_j} \frac{1}{l_j^{(i)}} > \frac{r_s}{l_s} \quad (s = 1, \dots, n),$$

$f \in B_{(p)}^l(E^n)$, then $f \in B_{(q)}^r(E^n)$ and the inequality

$$\|f\|_{B_{(q)}^r(E^n)} \leq C \|f\|_{B_{(p)}^l(E^n)} \quad (5)$$

holds.

Theorem 3. If $1 < p_i \leq q_j < \infty$, $0 \leq \alpha < p_1 / \max_i p_i'$ ($i = 1, \dots, k$; $j = 1, \dots, \tau$); $l_i = \bar{l}_i + \alpha_i$, $0 < \alpha_i \leq 1$ ($i = 1, \dots, n$);

$$1 - \sum_{i=1}^n \frac{1}{l_i} (1 + \nu_i) - \frac{\alpha}{p_1 l_n} + \sum_{i=1}^k \frac{1}{p_i'} \sum_{j=1}^{\tilde{n}_i} \frac{1}{l_{ij}} + \sum_{i=1}^{\tau} \frac{1}{q_i} \sum_{j=1}^{\tilde{m}_i} \frac{1}{l_j^{(i)}} > 0,$$

$f \in B_{(p),\alpha}^+(E^n)$, then $D_1^{\nu_1} \dots D_n^{\nu_n} f \in L_{(q)}(E^n)$ and the inequality

$$\|D_1^{\nu_1} \dots D_n^{\nu_n} f\|_{L_{(\mathbf{q})}(E^n)} \leq C \|f\|_{B_{(\mathbf{p}), \alpha}^l(E^n)} \quad (6)$$

holds.

Theorem 4. If $1 < p_i \leq q_j < \infty$, $0 \leq \alpha < p_1 / \max_i p'_i$ ($i = 1, \dots, k$; $j = 1, \dots, \tau$); $l_i = \bar{l}_i + \alpha_i$, $0 < \alpha_i \leq 1$, $r_i = \bar{r}_i + \beta_i$, $0 < \beta_i \leq 1$ ($i = 1, \dots, n$);

$$1 - \sum_{i=1}^n \frac{1}{l_i} - \frac{\alpha}{p_1 l_n} + \sum_{i=1}^k \frac{1}{p'_i} \sum_{j=1}^{\tilde{n}_i} \frac{1}{l_{ij}} + \sum_{i=1}^{\tau} \frac{1}{q_i} \sum_{j=1}^{\tilde{m}_i} \frac{1}{l_j^{(i)}} > \frac{r_s}{l_s} \quad (s = 1, \dots, n);$$

$f \in B_{(\mathbf{p}), \alpha}^l(E^n)$, then $f \in B_{(\mathbf{q})}^r(E^n)$ and the inequality

$$\|f\|_{B_{(\mathbf{q})}^r(E^n)} \leq C \|f\|_{B_{(\mathbf{p}), \alpha}^l(E^n)} \quad (7)$$

holds.

In inequalities (4)–(7) the constant C does not depend on f .

Let us indicate the scheme of proof, for example, of Theorem 2. Consider the norm $\|f\|_{\mathcal{L}_{(\mathbf{q})}^{r_s}(E^n)}$: its estimate, in view of Theorem 1, reduces to estimating the expression

$$N = \left(\int_0^{h^{\chi_s}} \frac{dz}{z^{1+q_\tau \beta_s}} \|\Delta_s^2(z) D_s^{\bar{r}_s} f(x)\|_{L_{(\mathbf{q})}(E^n)}^{q_\tau} \right)^{1/q_\tau}, \quad \text{where } \chi_s = \frac{1}{l_s} \quad (s = 1, \dots, n).$$

Using the integral identity of V. P. Il' in (3), and making the necessary transformations and estimates, we obtain

$$\begin{aligned} N &\leq C \left[\int_0^{h^{\chi_s}} \frac{dz}{z^{1+(\beta_s-2)q_\tau}} h^{(-\delta-\bar{r}_s \chi_s-2\chi_s)} \|f\|_{L_{(\mathbf{p})}(E^n)}^{q_\tau} \right]^{1/q_\tau} \\ &+ C \sum_{i=1}^n \left[\int_0^{h^{\chi_s}} \frac{dz}{z^{1+\beta_s q_\tau}} \times \left\| \int_0^{z^{1/\chi_s}} \frac{dv}{v^{1+\lambda_i-\varepsilon+\gamma_i \chi_i+r_s \chi_s}} \int_0^{v^{\chi_i}} t^{\gamma_i-(\frac{1}{p_k}+\alpha_i)} dt \int_x^{x+v^k} \Delta_i^2\left(\frac{t}{2}\right) D_i^{\bar{l}_i} f(y) dy \right\|_{L_{(\mathbf{q})}(E^n)}^{q_\tau} \right. \\ &+ C \sum_{i=1}^n \left[\int_0^{h^{\chi_s}} \frac{dz}{z^{1+(\beta_s-2)q_\tau}} \left\| \int_{z^{1/\chi_s}}^h \frac{dv}{v^{1+\lambda_i-\varepsilon+\bar{r}_s \chi_s+2\chi_s+\gamma_i \chi_i}} \int_0^{v^{\chi_i}} t^{\gamma_i-(1/p_k+\alpha_i)} dt \right. \right. \\ &\quad \left. \left. \times \int_x^{x+v^k} \Delta_i^2\left(\frac{t}{2}\right) D_i^{\bar{l}_i} f(y) dy \right\|_{L_{(\mathbf{q})}(E^n)}^{q_\tau} \right]^{1/q_\tau} = N_1 + N_2 + N_3, \end{aligned}$$

where

$$\delta = \sum_{i=1}^n \chi_i - \sum_{i=1}^k \frac{1}{p_k} \sum_{j=1}^{n_i} \chi_{ij} - \sum_{i=1}^{\tau} \frac{1}{q_i} \sum_{j=1}^{m_i} \chi_j^{(i)},$$

$$0 < \gamma_i \leq \frac{1}{p_k} + \alpha_i, \quad \lambda_i = \frac{\chi_i}{p_k} + \sum_{j=1}^n \chi_j - \delta, \quad \varepsilon = 1 - \delta.$$

Let us estimate one of the expressions, for example N_2 (N_1, N_3 are estimated analogously). For this we choose γ_i so that $\varepsilon - r_s \chi_s > \gamma_i \chi_i$ ($i = 1, \dots, n$; $s = 1, \dots, n$); then, after elementary estimates, we shall have

$$N_2 \leq ch^{\varepsilon - r_s \chi_s} \sum_{i=1}^n \left[\int_0^{h \chi_s} \frac{dz}{1 + \frac{q_\tau}{\chi_s} (\gamma_i \chi_i - \mu)} \left\| \left(\int_0^{z^{1/\chi_s}} \frac{dv}{1 + p_k \sum_{i=1}^{\tau} \frac{1}{q_i} \sum_{j=1}^{m_i} \chi_j^{(i)} + \mu p_k} \right)^{1/p_k} \right\|_{L^{(p)}[\square_v x_{n_1}(x_{n_1}), \dots, \square_v x_{n_k}(x_{n_k})]}^{p_k} \right]^{1/q_\tau} \times \int_0^{v \chi_i} t^{\chi_i p_k - (1 + p_k \alpha_i)} dt \left\| \Delta_i^2 \left(\frac{t}{2} \right) \bar{D}_i^{l_i} f(y) \right\|_{L^{(p)}(E^n)}^{p_k} \leq$$

(where μ is an arbitrary positive number); using the generalized Minkowski inequality and lemma (1) of A. Kh. Gudiev⁵ the required number of times, we shall have

$$\leq ch^{\varepsilon - r_s \chi_s} \sum_{i=1}^n \left[\int_0^{h \chi_s} \frac{dz}{1 + \frac{q_\tau}{\chi_s} (\gamma_i \chi_i - \mu)} \left(\int_0^{z^{1/\chi_s}} \frac{dv}{v^{1 + \mu p_k}} \int_0^{v \chi_i} t^{\chi_i p_k - (1 + p_k \alpha_i)} dt \right)^{q_\tau} \right]^{1/q_\tau} \times \left\| \Delta_i^2 \left(\frac{t}{2} \right) \bar{D}_i^{l_i} f(y) \right\|_{L^{(p)}(E^n)}^{p_k} \leq$$

changing the order of integration; choosing μ so that $\gamma_i \chi_i - \mu > 0$, we obtain

$$\leq ch^{\varepsilon - r_s \chi_s} \sum_{i=1}^n \|f\|_{L_{(p)}^{l_i}(E^n)}.$$

From the estimates obtained, for $h = 1$ we obtain inequality (5).

In conclusion I express my deep gratitude to Acad. S. L. Sobolev for his attention to this work, and also to A. Kh. Gudiev for posing the question and for valuable advice.

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Note: Figure translations are in progress. See original paper for figures.

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