

CLASSIFICATION AND CHARACTERISTIC PROPERTIES OF CARLEMAN OPERATORS

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Abstract

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MATHEMATICS

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CLASSIFICATION AND CHARACTERISTIC PROPERTIES OF CARLEMAN OPERATORS

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1°. Classification of Carleman operators.

Let (X, S, μ) be a space with a completely σ -finite measure (⁽¹⁾, p. 77) and let $K(s, t)$ be a $(\mu \times \mu)$ -measurable function defined on $X \times X$. Below we consider integral operators with kernels satisfying the following (not necessarily all*) conditions:

(I) T. Carleman's condition (^(2, 3))

$$\int_X |K(s, t)|^2 d\mu(t) < \infty$$

for μ -almost all (a.e.) $s \in X$.

(II) $K(s, t) = \overline{K(t, s)}$ for $(\mu \times \mu)$ -a.e. $(s, t) \in X \times X$.

(III) N. I. Akhiezer's condition (^(3, 4)): there exists a μ -measurable, μ -a.e. finite, nonnegative function $P(s)$, defined on X , such that

$$|K(s, t)| \leq P(s)P(t)$$

for $(\mu \times \mu)$ -a.e. $(s, t) \in X \times X$.

Functions satisfying conditions (I), (II) are called Carleman kernels (^(2, 3)) (abbreviated (C) -kernels). Functions satisfying condition (I) are called semi-Carleman kernels (⁽⁵⁾) ((SC) -kernels). Functions satisfying condition (III) are called B -kernels (⁽⁴⁾). A (C) -kernel satisfying condition (III) will be called a (BC) -kernel. An (SC) -kernel satisfying condition (III) will be called a (BSC) -kernel.

A densely defined integral operator acting in $L_2(X, S, \mu)$,

$$Lf = \int_X K(s, t)f(t) d\mu(t), \quad f \in D_L, \quad (1)$$

with $(\mu \times \mu)$ -measurable kernel $K(s, t)$, will be called: a (C) -operator, if $K(s, t)$ is a (C) -kernel; an (SC) -operator, if $K(s, t)$ is an (SC) -kernel; a (BC) -operator, if $K(s, t)$ is a (BC) -kernel; a (BSC) -operator, if $K(s, t)$ is a (BSC) -kernel.

As in ⁽⁶⁾, a densely defined linear operator T acting in $L_2(X, S, \mu)$ will be called an operator of type (C) if it is unitarily equivalent to a (C) -operator. Operators of types (SC) , (BC) , and (BSC) are defined analogously.

Finally, following ⁽⁶⁾, we shall call an operator T a strong (C) -operator if, for every unitary operator U in $L_2(X, S, \mu)$, the operator UTU^{-1} is a (C) -operator. Strong (SC) -, strong (BC) -, and strong (BSC) -operators are defined analogously.

In the present paper, for each of the classes (C) , (SC) , (BC) , (BSC) , the following three problems are considered**:

* But necessarily Carleman's condition (I). We call such operators Carleman operators.

** The first two problems for the class (C) were posed in ⁽⁷⁾. The third problem (for the class (SC)) was posed in ⁽⁶⁾ and studied in ^(6, 9).

Find necessary and sufficient conditions under which the operator T is: 1) an operator of the given class; 2) an operator of the given type; 3) a strong operator of the given class.

The paper gives solutions of each of the three problems for all four classes. The results obtained are, in a certain sense, final, with the exception of one case (BSC) —2)—the problem is solved only for normal operators.

Everywhere below in the paper it is assumed that (X, S, μ) is a separable space (⁽¹⁾, p. 165) with a completely σ -finite measure (⁽¹⁾, p. 77), and that T is a densely defined linear operator in $L_2(X, S, \mu)$.

Theorem 1. 1) The operator T is an (SC) -operator if and only if T has a majorant, i.e., if there exists a μ -measurable, μ -a.e. finite nonnegative function $\Lambda(s)$ defined on X such that for all $f \in D_T$

$$|(Tf)(s)| \leq \Lambda(s)\|f\| \quad \text{for } \mu\text{-a.e. } s \in X. \quad (2)$$

- 2) Suppose that the measure μ is not purely atomic*. The operator T is an operator of type (SC) if and only if the adjoint operator T^* is densely defined and the limiting spectrum of T^* contains 0.
- 3) Suppose that the measure μ is not purely atomic. The operator T is a strong (SC) -operator if and only if its closure is a Hilbert-Schmidt operator.

The second assertion of Theorem 1, in the case where T is a normal operator, $X = (a, b)$, and μ is Lebesgue measure, coincides with Theorem 1 of paper ⁽⁶⁾.

The third assertion of Theorem 1, in the case where $X = (a, b)$ and μ is Lebesgue measure, is a strengthening of Theorem 4 of paper ⁽⁶⁾ and Theorem 3 of paper ⁽⁹⁾.

Remark. Condition (2) of Theorem 1 is equivalent to the following conditions (⁽³⁾, p. 463):

$$D_{T^*} \supset [L_2]_\Lambda = \{f : f \in L_2(X, S, \mu), \quad \|f\|_\Lambda = \int \Lambda(s)|f(s)| d\mu(s) < \infty\} \quad (3)$$

$$\|T^*f\| \leq \|f\|_\Lambda \quad \text{for all } f \in [L_2]_\Lambda.$$

Theorem 2. 1) The operator T is a (C) -operator if and only if the operator T has a majorant $\Lambda(s)$ such that the inclusion (3) holds and**

$$(T^*f, g) = (f, T^*g) \quad \text{for all } f, g \in [L_2]_\Lambda. \quad (4)$$

- 2) Suppose that the measure μ is not purely atomic. The operator T is an operator of type (C) if and only if there exists a symmetric operator A such that $A \subseteq T^*$ and the limiting spectrum of the operator A contains 0.
- 3) Suppose that the measure μ is not purely atomic. The operator T is a strong (C) -operator if and only if its closure is a self-adjoint Hilbert-Schmidt operator.

The first assertions of Theorems 1 and 2 for the case $X = \Omega \subseteq R_n$, μ Lebesgue measure, were proved in ⁽⁸⁾. The second assertion of Theorem 2 is a generalization of a well-known theorem of J. von Neumann (⁽⁷⁾; ⁽³⁾, p. 467).

* We shall say that the measure μ is not purely atomic if in X there exists a set E of finite nonzero μ -measure such that, for any atom τ , $\mu(E \cap \tau) = 0$. It can be shown that if μ is purely atomic, then any bounded operator in $L_2(X, S, \mu)$ is a strong (BSC) and, consequently, a strong (SC) -operator.

** Condition (4) ensures the Hermiticity of the kernel and was first considered by N. I. Akhiezer (⁽⁴⁾, p. 129).

Theorem 3. 1) The operator T is a (BSC) -operator if and only if the operator T has a majorant $\Lambda(s)$ such that $D_{T^*} \supset [L_2]_\Lambda$ and, for all $f \in [L_2]_\Lambda$,

$$|(T^*f)(t)| \leq \Lambda(t)\|f\|_\Lambda \quad \text{for } \mu\text{-a.e. } t \in X, \quad (5)$$

where

$$\|f\|_\Lambda = \int_X \Lambda(s)|f(s)| d\mu(s).$$

- 2) Suppose that the measure μ is not purely atomic. A normal operator N is an operator of type (BSC) if and only if the residual spectrum of the adjoint operator N^* contains 0.
- 3) Suppose that the measure μ is not purely atomic. The operator T is a strong (BSC) -operator if and only if its closure is a nuclear operator.

Remark. Condition (5) of Theorem 3.1) is equivalent to the condition

$$|(T^*f, g)| \leq \|f\|_{\Lambda} \|g\|_{\Lambda}$$

for all $f, g \in [L_2]_{\Lambda}$, introduced by N. I. Akhiezer in the study of symmetric (BC) -operators ((⁴), p. 129).

Theorem 4. 1) The operator T is a (BC) -operator if and only if the conditions of Theorem 3.1) and condition (4) of Theorem 2.1) are satisfied.

- 2) Suppose that the measure μ is not purely atomic. An operator is an operator of type (BC) if and only if there exists a symmetric operator A such that $A \subseteq T^*$ and the residual spectrum of the operator A contains 0.
- 3) Suppose that the measure μ is not purely atomic. The operator T is a strong (BC) -operator if and only if its closure is a self-adjoint nuclear operator.

3°. Some generalizations of Theorem 1.

A. A densely defined linear integral operator L in $L_2(X, S, \mu)$, taking values in $L_2(X_0, S_0, \mu_0)$, will be called a Carleman operator if its kernel $K(s, t)$ is $(\mu_0 \times \mu)$ -measurable and

$$\int_X |K(s, t)|^2 d\mu(t) < \infty \quad \text{for } \mu_0\text{-a.e. } s \in X_0.$$

Theorem 5. 1) The operator τ is a Carleman operator if and only if τ has a majorant.

- 2) Suppose that (X_0, S_0, μ_0) is a separable space and that the measure μ_0 is not purely atomic. In order that there exist in $L_2(X_0, S_0, \mu_0)$ a unitary operator U such that $U\tau$ is a Carleman operator, it is necessary and sufficient that the adjoint operator τ^* be densely defined and that in the domain D_{τ^*} there exist an orthonormal sequence $\{f_n\}$ such that

$$\lim_{n \rightarrow \infty} \|\tau^* f_n\| = 0.$$

- 3) Suppose that (X_0, S_0, μ_0) is a separable space and that the measure μ_0 is not purely atomic. The operator $U\tau$ is a Carleman operator for every unitary operator U in $L_2(X_0, S_0, \mu_0)$ if and only if the closure of the operator τ is a Hilbert-Schmidt operator.

B. A densely defined linear operator L in H , taking values in $L_2(H_0; X_0, S_0, \mu_0)$ (H, H_0 are separable Hilbert spaces; for the definition of $L_2(H_0; X_0, S_0, \mu_0)$, see ⁽¹⁰⁾, p. 103), will be called Carleman if there exists a strongly μ_0 -measurable operator function $L(s)$ defined on X_0 (⁽¹⁰⁾, p. 88), taking values in the space of bounded linear operators acting from H to H_0 , such that $(Lh)(s) = L(s)h$ for μ_0 -a.e. $s \in X_0$ ($h \in D_L$). For such operators a theorem analogous to Theorem 5 is valid.

In conclusion, we note that the sets of full measure on which (1), (2), (5) are fulfilled depend on f .

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