

Mappings of Families of Sets

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Abstract

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MATHEMATICS

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MAPPINGS OF FAMILIES OF SETS

1. We are concerned with the following question, which was considered in the preceding article ⁽¹⁾. Let A_n, A'_m be two affine spaces, T, T' the groups of their translations, and let M be a bounded set in A_n . Let f be a one-to-one mapping of A_n onto A'_m such that for every $t \in T$ there exists a $t' \in T'$ such that

$$ft(M) = t'f(M), \quad (1)$$

and conversely: for every $t' \in T'$ there exists a $t \in T$ for which (1) is satisfied. The question is what such mappings can be. In ⁽¹⁾ an answer to this question was given for a broad class of sets M , but we were forced to omit the proofs. In the present communication brief proofs are given of more particular results, which include, however, the cases when M is a convex or an arbitrary smooth domain.

2. In ⁽¹⁾ the following construction was given. Let O be a point in A_n . Putting $M = M_0$, we define the sets M_i inductively:

$$M_i = \bigcup t(M_{i-1}), \quad O \in t(M_{i-1}). \quad (2)$$

These sets (for $i \geq 1$) possess the following properties.

(I). If M_{iX} is constructed about the point X in the same way as M_i is about O , then $M_{iX} = t(M_i)$, where t is the translation $O \rightarrow X$.

(II) O is a center of symmetry of M_i . The proof is obvious.

(III) Let M'_i ($i = 1, 2, \dots$) be the sets constructed in A'_m about the point $O' = f(O)$, starting from the set $M' = f(M)$, in the same way as the M_i are constructed from M . Then $M'_i = f(M_i)$, and in general, for $X' = f(X)$,

$$M'_{iX'} = f(M_{iX}). \quad (3)$$

This follows from property (1) of the mapping f . If t is the translation $O \rightarrow X$, and t' is the translation $O' = f(O) \rightarrow X' = f(X)$, then (3) is equivalent to

$$f(M_{iX}) = ft(M_i) = t'f(M_{iX}) = M'_{iX'}, \quad t'f(M_i) = ft(M_i). \quad (4)$$

Thus, for the set M_i , condition (1) is satisfied with the additional property that the center $t(M_i)$ is mapped to the center $t'f(M_i)$. Therefore, instead of the given M , one may consider any of the M_i .

3. We formulate the theorems to be proved. In Theorems 1-3 it is assumed that f is continuous, and therefore is a homeomorphism of A_n onto A'_m (so that $m = n$). Moreover, everywhere M is understood to mean one of the M_i , and $M' = f(M)$ is bounded, while $n \geq 2$. A point X of a set N will be called extreme from within if it is a limit point of its interior and is not contained in any simplex with vertices in N distinct from X . At such a point N has a supporting plane, and if it is tangent to the set N at such a point X , i.e. is a contingency of its boundary ∂N at X , then it is the unique supporting plane to N at the point X .

Theorem 1. *If M has an extreme-from-within point and at it a tangent supporting plane P_0 , then $f(P_0)$ is an $(n-1)$ -dimensional plane in A'_n . For every plane $P \parallel P_0$, $f(P)$ is a plane parallel to $f(P_0)$.*

It follows immediately from Theorem 1 that if M has $(n+1)$ tangent planes in general position at points that are extreme from within, then f is affine. But from it one also derives the stronger assertion that f is affine if there are only n such planes.

4. A convex body may have no tangent planes at any extreme point, and then Theorem 1 gives nothing. However, the following holds.

Theorem 2. *If M is a convex body, but not a cylinder, then f is affine.*

A cylinder C is the Cartesian product of a segment and a flat set. In general,

$$C = l_1 \times \dots \times l_k \times N, \quad (5)$$

where the l_i are segments, and N is contained in an $(n-k)$ -dimensional plane and is no longer decomposable into the product of a segment and a set of smaller dimension. In the extreme case $C = l_1 \times \dots \times l_n$ is a parallelepiped, and then in (5) N is understood to be a point. To each l_j there corresponds a base plane P_j , spanned by the remaining segments l_i and the set N .

Let P be an $(n-1)$ -dimensional plane in A_n and let l be a vector not parallel to P . By a shear d_{Pl} we mean such a homeomorphism of A_n onto itself which is a parallel translation on every plane parallel to P , and sends every segment equal and parallel to l into a segment again equal and parallel to l . The segments l_i in formula (5) may be regarded as vectors. Then it is clear that every shear $d = d_{l_i P_i}$ (where P_i is the base plane) is equivalent to some translation of the cylinder C , i.e. $d(C) = t(C)$. Comparing this with (1), we see that every shear

$d_{l_i P_i}$ is a mapping f of A_n onto itself when $M = C$. At the same time, the following holds.

Theorem 3. *If M is a convex cylinder, then*

$$f = f_0 d_1 \dots d_k, \quad (6)$$

where f_0 is an affine mapping of A_n onto A'_n , and the d_i are arbitrary shears $d_{l_i P_i}$. Moreover, they can be normalized so that $d_i(M) = M$, so that one obtains $f(M) = f_0(M)$, i.e. $f(M)$ is necessarily an affine image of M .

If the original M is convex, then every M_i is convex. The converse is, of course, false. The most important case is when M is the boundary of a convex body. Then M_1 will, as is easy to verify, be a convex body. Therefore, from Theorems 2, 3 it follows that if M is the boundary of a convex body H and H is not a cylinder, then f is affine; if, however, H is a cylinder, then (6) holds.

5. Concerning the continuity of f , we shall prove the following theorem.

Theorem 4. *If M is open or closed with interior points and f^{-1} is bounded, then f is continuous.*

6. We shall prove Theorem 1. Let M have an extreme interior point B and, at it, a tangent plane P_0 . By central symmetry of M there is a symmetric extreme point $\widetilde{B}_0$ and, at it, a tangent plane $\widetilde{P}_0 \parallel P_0$. From the construction (2) of the sets M_i it is easy to conclude that the same is true for each M_i : it has a pair of symmetric extreme interior points $B_i, \widetilde{B}_i$ with tangent planes $P_i, \widetilde{P}_i$, parallel to P_0 .

Take a plane $P \parallel P_0$ and a point $X \in P$. Subject each M_i to two translations: $B_i \rightarrow X$ and $\widetilde{B}_i \rightarrow X$. We obtain sets $t(M_i), \tilde{t}(M_i)$ with common point X and common tangent supporting plane P . Here $t(M_i)$ and $\tilde{t}(M_i)$ lie on different sides of P and have no other common points except X , otherwise X would not be extreme for them.

The sets $t(M_i)$ and $\tilde{t}(M_i)$ are mutually symmetric with respect to X , since they are centrally symmetric, situated in parallel fashion, and have only one common point X . Therefore their sums N and $\widetilde{N}$ are also mutually symmetric with respect to X . Here $N \setminus (X)$ and $\widetilde{N} \setminus (X)$ are open half-spaces bounded by the plane P (as follows from the fact that X is a limiting point for interior points of all the sets $t(M_i), \tilde{t}(M_i)$, while the plane P is tangent to their boundaries). Therefore

$$P = [A_n \setminus (N \cup \widetilde{N})] \cup (X).$$

Now let $P' = f(P)$, $X' = f(X)$, $N' = f(N)$, $\widetilde{N}' = f(\widetilde{N})$. The sets $N', \widetilde{N}'$ are sums of f -images of the sets $t(M_i), \tilde{t}(M_i)$. But by virtue of the conditions

by (1) these images are translated sets $M'_i = f(M_i)$. They are centrally symmetric and pairwise have a single common point X' . Therefore they are pairwise symmetric with respect to X' , and hence N' and $\tilde{N}'$ are also symmetric with respect to X' .

The complement of the set $(N' \cup \tilde{N}') \setminus (X')$ is the f -image of the plane P . Therefore $f(P)$ is also symmetric with respect to X' . But the point X was chosen arbitrarily on P , so the set $f(P)$ is symmetric with respect to each of its points. And since it is homeomorphic to P , it is itself an $(n-1)$ -dimensional plane, and Theorem 1 is proved.

7. Let us turn to the proof of Theorems 2 and 3. Let M be a convex body, $\bar{\Gamma}$ a face of M , i.e. the intersection of M with some supporting plane P , and let Q be the plane spanned by $\bar{\Gamma}$. Replacing M by M_i , we may assume that M is centrally symmetric. We shall prove that $f(Q)$ is a plane.

If $\bar{\Gamma}$ is a point, the assertion is trivial. Therefore suppose that

$$k = \dim \bar{\Gamma} > 0,$$

and call the face Γ the interior of $\bar{\Gamma}$ relative to Q . Let $\tilde{\Gamma}$ and $\tilde{P}$ be the face and supporting plane symmetric to Γ and P . Subject M to such a translation t_0 that $t_0(\tilde{P}) = P$ and $t_0(\tilde{\Gamma})$ intersects Γ .

Taking a small k -dimensional ball $V \subset \tilde{\Gamma} \cap t_0(\Gamma)$, we may subject M to arbitrary small translations t along Q such that $t(M) \supset V$. In other words, in the group T_Q of translations parallel to Q , there is a neighborhood of zero U such that, for $t \in U$, $t(\Gamma) \supset V$. Under the mapping f we obtain the sets

$$ft_0(M) = t'_0(M')$$

and

$$ft(M) = t'(M').$$

Since f is a homeomorphism, $t'_0(M') \cap t'(M')$ are k -dimensional, just as $t_0(M) \cap t(M)$.

The homeomorphism f defines a homeomorphism φ of the group T onto T' :

$$\varphi(t) = t',$$

where t' corresponds to t by virtue of relation (1), and moreover $\varphi(0) = 0$. Therefore the set $\varphi(U)$ is a neighborhood of zero of the group T' relative to $\varphi(T_Q)$. To the translations $t \in U$ for which $t(\Gamma) \supset V$, there correspond translations t' for which $ft(\Gamma)$ covers $f(V) \subset f(\bar{\Gamma})$. Such sufficiently small translations form a local group—a neighborhood of zero in some subgroup of the entire group T' .

It follows that, at least near zero, $\varphi(U)$ is such a local group. This means that $f(\Gamma)$ admits a locally free parallel displacement along $f(V)$. Consequently, $f(V)$ is a domain in a k -dimensional plane.

But since we could choose the translation t_0 so that $t_0(\tilde{\Gamma})$ covered a neighborhood V of any given point in Γ , it follows that $f(\Gamma)$ is an open set in a k -dimensional plane. Translating M and $t_0(M)$ along the plane Q , we see that $f(Q)$ itself is a plane of the same dimension.

8. Under the assumptions of item 7, let us prove that if P is a plane parallel to a tangent plane to M , then $f(P)$ is also a plane.

Let P_0 touch M along the face Γ , and let Q_0 be the plane of this face. Let $P \parallel P_0$ and $x \in P$. Subject each set M_i to all such translations that its corresponding face falls on Q_0 , and also to translations under which the face symmetric to it also falls on Q_0 . Then the sums of such sets over all i form two sets $N, \tilde{N}$, with

$$N \cap \tilde{N} = Q,$$

and $N, \tilde{N}$ mutually symmetric with respect to every point on Q and, in particular, the point X , while $N \setminus Q, \tilde{N} \setminus Q$ are open half-spaces bounded by P . Now, applying the mapping f and using the fact that $f(Q)$ is also a k -dimensional plane, we see that $f(N)$ and $f(\tilde{N})$ are mutually symmetric with respect to $f(X)$. Hence, just as in the proof of Theorem 1, we conclude that $f(P)$ is an $(n-1)$ -dimensional plane.

9. **Proof of Theorem 2.** If a convex body is not a cylinder, then it has no fewer than $(n+1)$ tangent planes in general position. Therefore Theorem 3 follows from what was proved in item 8.
10. **Proof of Theorem 3.** Let the convex body M be the cylinder (5). It has faces parallel to each generator l_i . Therefore, by what was proved in § 7, the lines $L_i \parallel l_i$ are mapped into lines. The set N is a face of M , and therefore the plane Q spanned by it is mapped into a plane; moreover, since N is not a cylinder, by Theorem 2, f is affine on Q and, of course, on all such parallel planes. From these observations we easily arrive at Theorem 3.
11. Let us prove the first part of Theorem 4, when M_i is open. We shall write M instead of M_i . Fix a point $X \in A_n$, and let $X' \in f(X)$, while U' is a neighborhood of X' . Let H be the closed convex hull of M' . Since M' is centrally symmetric, so is H . Let $B, \tilde{B}$ be its mutually symmetric extreme points. It is necessary that $B, \tilde{B} \in \overline{M'_1}$.

Let $t', \tilde{t}'$ be translations: $B \rightarrow X', \tilde{B} \rightarrow X'$. Then, since the points $B, \tilde{B}$ are extreme, $t'(\overline{M'}) \cap \tilde{t}'(\overline{M'}) = (X')$. In view of this, there are such small translations $\tau, \tilde{\tau}$ that

$$X' \in \tau t'(M') \cap \tilde{\tau} \tilde{t}'(M) \subset U'. \quad (7)$$

To these translations $\tau t', \tilde{\tau} \tilde{t}'$, by virtue of (1), there correspond such translations $t, \tilde{t}$ that

$\tau t'(M') = ft(M)$ and similarly for $\tilde{\tau} \tilde{t}'$. Therefore from (7) it follows (since f is one-to-one)

$$f(X) \in f(t(M) \cap \tilde{t}(M)) \subset U'. \quad (8)$$

But M is open, and therefore $t(M) \cap \tilde{t}(M)$ is a neighborhood of X . Thus the continuity of f has been proved.

12. Let us prove the second part of Theorem 4, when M is closed with interior points and f^{-1} is bounded. Since M has interior points, the sets M_i together cover all of A_n . Therefore their f -images M'_i cover A'_n . If, moreover, no M'_j covered a given bounded domain, then f^{-1} would not be bounded. Consequently, for some j , M'_j has interior points. For brevity denote this M'_j by M' and M_j by M .

Take any point $X' \in A_{m'}$ and about it an open ball U so small that it is covered by some set $t'(M')$. Let r be the radius of U . Form the set V , the intersection of all $t'(M') \supset U$. It contains no balls equal to U (except U itself), since if U' is such a ball and τ is the translation from its center to the center of U , then $\tau(V) \supset U$ and, consequently, by the definition of the set V , it must be that $\tau(V) = V$. But then V is unbounded, contrary to the boundedness of M' .

Thus, V contains no balls equal to U . Hence it follows that outside the ball $2U$, concentric with U and of double radius, the set V contains no open balls U' of radii $r' \geq r - \varepsilon$ with some $\varepsilon > 0$.

Take a ball U_1 of radius $r_1 = \varepsilon/2$, concentric with U . Define from it the set V_1 , as V is defined from U . Then, in view of the choice of r_1 , it turns out that $V_1 \subset 2U$. Continuing this process, we obtain a sequence of sets V_i :

$$U_i \subset V_i \subset 2U_{i-1}, \quad \bigcap V_i = (X').$$

Now observe that, since V_i is the intersection of all $t'(M') \supset U_i$, each $f^{-1}(V_i)$ is the intersection of all $f^{-1}t'(M') \supset f^{-1}(U_i)$. By relation (1), $f^{-1}t'(M) = t(M)$, and, by assumption, M is closed and bounded, i.e. compact. Therefore the sets $f^{-1}(V_i)$ are compact. But their intersection is the point $X = f^{-1}(X')$. Therefore, whatever neighborhood W of the point X is chosen, there is an i such that $f^{-1}(V_i) \subset W$. This proves the continuity of f^{-1} , and hence also of f .

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