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Abstract

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MATHEMATICS

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ON THE EXISTENCE OF Π -COMPLEMENTS TO NORMAL SUBGROUPS OF FINITE GROUPS

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In a survey lecture at the Edinburgh Mathematical Congress ⁽¹⁾, Wielandt drew attention to the importance of eliminating the commutativity condition on the normal subgroup in the following theorem of Gaschütz ⁽²⁾: *a normal abelian subgroup K of a group G has a complement in G , if for every prime p there exists a complement to $G_p \cap K$ in G_p* . In the present paper, continuing our investigations on the theory of complements ⁽³⁾, we obtain a solution of this question (Theorem 2).

We consider only finite groups. Notation: Π is some set of primes; G is a finite group of order $|G|$; $O^\Pi(G)$ is the subgroup of the group G generated by all those elements of G whose orders are not divisible by numbers from Π ; G_p is one of the Sylow p -subgroups of the group G , p is prime.

We shall say that a subgroup K of a group G has a Π -complement H in G if $HK = G$ and $|H \cap K|$ is not divisible by numbers from Π ; a subgroup H with this property will be called a Π -complement to (for) K in G . If Π is the set of all prime numbers, then a Π -complement becomes an ordinary complement.

Theorem 1. *A normal subgroup K of a group G has a Π -complement in G , if for every $p \in \Pi$ the subgroup $G_p \cap K$ is abelian and complemented in G_p .*

Before proceeding to the proof of Theorem 1, we note some of its applications.

Theorem 2. *A normal subgroup K of a group G has a complement in G , if for every prime divisor p of the index $|G : K|$ the subgroup $G_p \cap K$ is abelian and complemented in G_p .*

Proof. Let Π be the set of all prime divisors of the index $|G : K|$ of the subgroup K . By Theorem 1, there exists a Π -complement H to the subgroup K in G . Since $|H \cap K|$ is not divisible by numbers from Π , while $H/H \cap K$ is a Π -group, there exists a complement to $H \cap K$ in H , which is the required complement for K .

Theorem 2 contains both Gaschütz's theorem and the Schur-Zassenhaus theorem on the existence of a complement to a Hall normal subgroup. The following theorem also generalizes Gaschütz's result.

Theorem 3. *A normal subgroup K of a group G has a complement in G , if for every prime p a Sylow p -subgroup of K is abelian and complemented in a Sylow p -subgroup of G containing it.*

Theorem 4. *A normal subgroup K of a group G has a Π -complement in G , if for every $p \in \Pi$ a Sylow p -subgroup of K is abelian and $O^p(K) = K$.*

Proof. Let P be a Sylow p -subgroup of the group G , $p \in \Pi$. According to a theorem of Gaschütz ((⁵), p.426), $P \cap K$ has a complement in P . It remains to apply Theorem 1.

For $K = O^\Pi(G)$, Theorem 4 yields a result giving an answer to the question posed in (³) (see also (⁴), problem 3.59).

Theorem 5. *If, for every $p \in \Pi$, the Sylow p -subgroup of $O^\Pi(G)$ is abelian, then $O^\Pi(G)$ has a complement in G .*

An extension of a group G is a group Γ containing a normal subgroup N isomorphic to G . The extension Γ is split if N has a complement in Γ . If Γ/N is a Π -group, then Γ is called an extension of G by means of a Π -group.

Theorem 6. *Let the group G coincide with its commutator subgroup and have an abelian Sylow p -subgroup for every $p \in \Pi$. Then every extension of the group G by means of a Π -group is split.*

We proceed to the proof of Theorem 1. Suppose that there exist groups for which Theorem 1 is not fulfilled. Choose among them a group G of least order. Thus, there will be a set Π of primes and a normal subgroup K of the group G such that there is no Π -complement to K in G . The set Π is nonempty, since in the case of empty Π the required Π -complement could be G itself. For the same reason Π contains at least one prime divisor of the order of K . For convenience we shall assume that every number in Π divides the order of G .

Now take such a subset ω of the set Π for which the following conditions are satisfied: 1) there is no ω -complement to the subgroup K in G ; 2) the subgroup K has an ω_1 -complement in G for every nontrivial subset ω_1 of the set ω . The set ω , obviously, is nonempty and contains at least one prime divisor of $|K|$. Let p be some element of ω dividing $|K|$. Denote by τ the set obtained from ω by deleting the element p . Then K has a τ -complement H in the group G .

Since $HK = G$, in H there is a subgroup M such that $MK = G$, but $M_1K \neq G$ for every nontrivial subgroup M_1 of M (such a subgroup M was called in [3] an addition to K in G). It is not difficult to see that $M \cap K$ lies in the Frattini subgroup of the group M , and hence is nilpotent. Since $M \cap K$ is contained in $H \cap K$, the order $|M \cap K|$ is not divisible by the numbers in τ , i.e. M is a τ -complement to K in G . Note also that p divides $|M \cap K|$, since K has no ω -complements.

In what follows M_p denotes some (fixed) Sylow p -subgroup of M , and G_p one of the Sylow p -subgroups of the group G that contains M_p . Obviously, $P = G_p \cap K$ is a Sylow p -subgroup in K . Moreover, $G_p = M_{pP}$, since $|MK : M_{pP}|$ is not divisible by p . Denote by P_1 the intersection

$$M_p \cap (M \cap K) = M_p \cap K = M_p \cap G_p \cap K = M_p \cap P.$$

It is clear that P_1 is a Sylow p -subgroup of $M \cap K$. The order of $M \cap K$ is divisible by p , therefore P_1 is distinct from the identity. Consider the normalizer N of the subgroup P_1 in the group G . We shall distinguish two cases.

First case. $N \neq G$. The subgroup N contains M , since $M \cap K$ is nilpotent and normal in M . Since P is abelian by hypothesis, P is also contained in N . Thus $G_p = M_{pP}$ is contained in N , and P , being Sylow in K and in $N \cap K$, has a complement in G_p by hypothesis. Let $q \in \omega$, $q \neq p$. Take a Sylow q -subgroup N_q of N containing a Sylow q -subgroup M_q of M . Then $N_q = M_{qQ}$, where $Q = N_q \cap K = N_q \cap (N \cap K)$ is a Sylow q -subgroup of $N \cap K$. The intersection $M_q \cap Q$ is equal to the identity, since $M_q \cap Q$ is contained in the subgroup $M \cap K$, whose order is not divisible by q . Thus we see that for N , $N \cap K$, and the set ω all the conditions are fulfilled. Therefore, in view of $|N| < |G|$, $N \cap K$ has an ω -complement S in N . From $S(N \cap K) = N$ it follows that $SK = G$, since $MK = G$ and $N \supseteq M$. The order of $S \cap K$ is not divisible by the numbers in ω , since $S \cap (N \cap K) = (S \cap K) \cap N = S \cap K$. Hence S is an ω -complement to K in G . We have arrived at a contradiction.

Second case. $N = G$, i.e. the subgroup P_1 is normal in G . Denote by R the product of all those normal subgroups of the group K each of which has no composition factors of order p . Obviously, the subgroup R is characteristic in K and also does not ...

has composition factors of order p . Since the Sylow p -subgroup P of K is abelian by assumption, applying Theorem 3.3 of [6] and the theorem of P. Hall and G. Higman ([5], p. 691), we see that PR/R is a normal Sylow p -subgroup of the group K/R . By assumption $G_p = P\bar{P}$, $P \cap \bar{P} = 1$. Since $G_p \cap R = P \cap R$ is a Sylow subgroup in R , it follows that $\bar{P} \cap R = 1$, and hence $PR/R \cap \bar{P}R/R = R/R$. Thus $\bar{P}R/R$ is a complement to PR/R in G_{pR}/R . By Gaschütz' s theorem ([5], p. 121), there exists a complement U/R to the subgroup PR/R in the group MPR/R .

Let $q \in \omega$, $q \neq p$. Since $|M \cap PR|$ is not divisible by q , a Sylow q -subgroup Q of MPR can be represented in the form $Q = M_q Q_1$, where Q_1 is a Sylow q -subgroup of R , intersecting M_q trivially. But the Sylow q -subgroups of MPR and U have the same orders, and hence $Q^x \leq U$ for some $x \in MPR$. Thus the Sylow q -subgroup of R has a complement in a Sylow q -subgroup of U containing it. Further, $O^p(R) = R$, since, by construction, R has no composition factors of order p . Therefore, by Gaschütz' s theorem ([5], p. 426), a Sylow p -subgroup of R has a complement in a Sylow p -subgroup of U containing it. Thus the hypothesis of the theorem is satisfied for the set ω , the group U , and its normal

subgroup R . Since $|U| < |G|$, the theorem is true for U . Hence R has an ω -complement L in the group U . The subgroup L will be an ω -complement to K in G . Indeed, from the three equalities $LR = U$, $UPR = MPR$, $MK = G$, it follows that $LK = G$. Moreover, it is not difficult to note that for any $q \in \omega$ the order of a Sylow q -subgroup of L is equal to the order of a Sylow q -subgroup of the factor group G/K . Therefore $|L \cap K|$ is not divisible by numbers from ω . We have again arrived at a contradiction.

Theorem 1 is proved.

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