

ON A CONCAVE $\backslash(n\backslash)$ -PERSON GAME AND ONE MODEL OF PRODUCTION

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Abstract

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MATHEMATICS

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ON A CONCAVE n -PERSON GAME AND ONE MODEL OF PRODUCTION

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In the present paper a method is given for finding a normalized equilibrium point of a concave n -person game, and a connection is established between this problem and A. Wald's classical model of production. A dynamic variant of this model is also considered, and a numerical method is constructed for finding its equilibrium state.

1. Let there be given functions $\varphi_i(x) \equiv \varphi_i(x_1, \dots, x_i, \dots, x_n)$ ($i = 1, \dots, n$), concave in $x_i \in E^{m_i}$ for fixed $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$, and continuous in $x = (x_1, \dots, x_i, \dots, x_n)$, and a convex compact set Ω in the $m = m_1 + \dots + m_n$ -dimensional space of vectors $x = (x_1, \dots, x_i, \dots, x_n)$.

Denote

$$\Phi(x, y) = \sum_{i=1}^n \varphi_i(x_1, \dots, y_i, \dots, x_n).$$

A point $x^* \in \Omega$ for which the condition

$$\Phi(x^*, x^*) = \max\{\Phi(x^*, x) \mid x \in \Omega\}, \quad (1)$$

is satisfied is called a normalized equilibrium point of the concave n -person game with payoff functions $\varphi_i(x_1, \dots, x_i, \dots, x_n)$ under strategies $(x_1, \dots, x_i, \dots, x_n) \in \Omega$.

The assumption of strict increase of the vector function

$$g(x) = (\partial\varphi_1(x)/\partial x_{11}, \dots, \partial\varphi_1(x)/\partial x_{1m_1}, \dots, \partial\varphi_n(x)/\partial x_{n1}, \dots, \partial\varphi_n(x)/\partial x_{nm_n}),$$

i.e., the requirement $(g(x) - g(y), y - x) > 0$, $\forall (x, y) \in \Omega \times \Omega$ ($x \neq y$), which ensures the uniqueness of the vector x^* , will be replaced by the conditions:

$$1) \Omega \text{ is a strictly* convex compact set; } 2) \|g(x)\| > 0, \forall x \in \Omega. \quad (2)$$

Denote

$$F(x) = \max\{(g(x), y - x) \mid y \in \Omega\} = (g(x), \theta(x) - x) = (g(x), \xi(x)).$$

Since problem (1) is a convex programming problem, and

$$g(x^*) = \nabla \Phi(x^*, y) \big|_{y=x^*},$$

we have $F(x^*) = 0$ (see, for example, (2)).

Below, for finding a normalized equilibrium point, a numerical method is constructed which is a certain generalization of the conditional-gradient method (2,3).

2. As an initial approximation we choose an arbitrary vector $x^0 \in \Omega$. Suppose that k steps of the algorithm have already been carried out and approximations $x^1, \dots, x^k$ have been constructed. To construct the $(k+1)$ -st approximation, we solve the following convex programming problem:

$$\max\{(g(x^k), x - x^k) \mid x \in \Omega\} = F(x^k) = (g(x^k), \theta(x^k) - x^k)$$

and compute the new approximation by the formula

$$x^{k+1} = x^k + t_k(\theta(x^k) - x^k) = x^k + t_k \xi^k,$$

where t_k may be chosen in one of the following ways, corresponding to three algorithms A1, A2, A3:

* The set Ω is strictly convex if from $x, y \in \Omega$ it follows that $\lambda x + (1 - \lambda)y \in \text{int } \Omega$, $\forall 0 < \lambda < 1$.

1°. As t_k one takes the k -th term of an arbitrary sequence $\{t_k\}$ satisfying only the conditions: a) $t_k \rightarrow 0$, $0 < t_k \leq 1$; b)

$$\sum_{k=0}^{\infty} t_k = \infty.$$

2°.

$$t_k = \max\{t \mid F(x^k + t\xi^k) \leq F(x^k)(1 - 1/2t), 0 < t \leq 1\}.$$

3°.

$$t_k = \max\{t = 2^{-\nu} \mid F(x^k + 2^{-\nu}\xi^k) \leq F(x^k)(1 - 2^{-(\nu+1)}), \nu \text{ is an integer}\}.$$

3. Theorem 1. Suppose condition (2) is satisfied and $\nabla F(x)$ is a continuous vector function. Then the following assertions hold:

1°. If for the Jacobian $H(x)$ of the vector function $g(x)$ there exists $\mu > 0$ such that

$$(H(x)\xi, \xi) \leq -\mu(\xi, \xi), \quad \forall x \in \Omega, \quad \forall \xi \in E^m, \quad (3)$$

then the sequence $\{x^k\}$ generated by algorithm A1 contains a subsequence $\{x^{k_i}\}$ converging to x^* .

2°. If $(H(x)\xi, \xi) \leq 0, \forall x \in \Omega$ and $\forall \xi \in E^m$, then the sequence $\{x^k\}$ generated by algorithm A2 converges to x^* .

3°. If condition (3) is satisfied and $\nabla F(x)$ satisfies the Lipschitz condition, then the entire sequence $\{x^k\}$ generated by algorithm A1 converges to x^* ; but if it is generated by algorithm A3, then

$$F(x^k) \leq F(x^0)q^k, \quad \|x^k - x^*\|^2 \leq \frac{F(x^0)}{\mu} q^k \quad (q < 1).$$

A subsequence converging to x^* , whose existence is guaranteed in 1°, can be constructed as follows:

$$k_1 = 0, \quad k_{i+1} = \min\{k \mid k > k_i, F(x^k) < F(x^{k_i})\}.$$

In algorithm A3, to find the value of t_k it may be necessary at each step to check the inequality many times:

$$F(x^k + 2^{-\nu}\xi^k) \leq F(x^k)(1 - 2^{-(\nu+1)}).$$

Since computing the value $F(x)$ is associated with solving an extremal problem, such a method may turn out to be uneconomical, and it can be replaced by the following:

$$t_k = \max\{t = t_{k-1}2^{-\nu} \mid F(x^k + t_{k-1}2^{-\nu}\xi^k) \leq F(x^k)(1 - t_{k-1}2^{-(\nu+1)}), \nu \text{ is an integer}\}.$$

Then, starting from some index k , the parameter t_k will not change, and to determine it one will need to compute the function $F(x)$ only at a single point $x^k + t_{k-1}\xi^k$.

Let us now pass to the consideration of two production models which are, in essence, certain concave n -person games.

4. First we consider the classical production model of A. Wald (⁴, ⁵). Suppose that in some economy n products are produced and r resources are used, available in quantities respectively $a_1, \dots, a_r$. The quantity a_{ij} ($i = 1, \dots, n; j = 1, \dots, r$) indicates the expenditure of the j -th resource necessary for producing one unit of the i -th product. However, unlike the classical problem of maximum profitability of enterprises (⁶), the prices of the products produced are not constant, but depend on the quantity of products produced. Let the function $f_i(x) \equiv f_i(x_1, \dots, x_n)$ ($i = 1, \dots, n$) indicate the price of a unit of the i -th product under the condition that the products are produced in quantities $x_1, \dots, x_i, \dots, x_n$. In (⁴) A. Wald, under certain assumptions concerning the functions $f_i(x)$ and under the assumption that

$$\text{the set } \Omega_0 = \left\{ x \mid \sum_{i=1}^n a_{ij}x_i \leq a_j \ (j = 1, \dots, r), \ x_i \geq 0 \ (i = 1, \dots, n) \right\}$$

is a polyhedron, answered affirmatively the question of the existence of a non-negative production vector $x^* = (x_1^*, \dots, x_i^*, \dots, x_n^*)$ and a nonnegative the vector of resource valuations $v^* = (v_1^*, \dots, v_r^*)$ such that

$$\sum_{i=1}^n a_{ij}x_i^* \leq a_j \quad (j = 1, \dots, r); \quad \sum_{j=1}^r a_{ij}v_j^* \geq f_i(x_1^*, \dots, x_i^*, \dots, x_n^*) \quad (i = 1, \dots, n),$$

$$\left(a_j - \sum_{i=1}^n a_{ij}x_i^* \right) v_j^* = 0 \quad (j = 1, \dots, r); \quad \left(\sum_{j=1}^r a_{ij}v_j^* - f_i(x^*) \right) x_i^* = 0 \quad (i = 1, \dots, n). \quad (4)$$

In [5] G. Kuhn, regarding the $f_i(x)$ as merely nonnegative and continuous for all $x \geq 0$, proved the existence of a production vector $x^* \in \Omega_0$ that is a solution of the following linear programming problem:

$$\max \left\{ \sum_{i=1}^n f_i(x^*)x_i^* \mid x \in \Omega_0 \right\} = \sum_{i=1}^n f_i(x^*)x_i^*. \quad (5)$$

The existence of vectors x^* and v^* satisfying conditions (4) is a simple consequence of equality (5) and the duality theorems of linear programming. It turns out that the vector x^* establishing equilibrium in Walras's model is a normalized equilibrium point of the following concave n -person game.

Let

$$\varphi_i(x_1, \dots, x_i, \dots, x_n) = \int_0^{x_i} f_i(x_1, \dots, y, \dots, x_n) dy$$

be the payoff of the i -th player, under the condition that the players have chosen, as their strategies, respectively, the values $x_1, \dots, x_i, \dots, x_n$.

Consider the problem of finding a normalized equilibrium point, i.e., a point $x^* = (x_1^*, \dots, x_i^*, \dots, x_n^*)$, for which

$$\Phi(x^*, x^*) = \sum_{i=1}^n \varphi_i(x_i^*) =$$

$$= \max \left\{ \sum_{i=1}^n \varphi_i(x_1^*, \dots, y, \dots, x_n^*) \mid (y_1, \dots, y_i, \dots, y_n) \in \Omega_0 \right\}. \quad (6)$$

Theorem 2. *If each function $f_i(x)$ ($i = 1, \dots, n$) is jointly continuous in the variables and does not increase in the corresponding variable x_i , and Ω_0 is a*

polyhedron, then, in order that the point x^* be a solution of problem (5), it is necessary and sufficient that this point be a solution of problem (6).

The established equivalence of A. Wald' s model to a concave n -person game makes it possible to apply, in finding an equilibrium state in this model, the numerical methods constructed in [7]. The algorithms constructed in Sec. 2 can also be applied to finding an equilibrium state in the dynamic variant of A. Wald' s model presented below.

5. Consider an economy in which n products are produced and r types of resources are used. We shall assume that the rate at which the output of each product changes at time t depends linearly on the quantity of each type of product produced at that time and on the quantities of resources used at that time. Therefore one may assume that the economy under consideration is described by the system of differential equations

$$dx/dt = A(t)x + B(t)u, \quad x(0) = x_0, \quad (7)$$

where the elements $a_{ij}(t)$ and $b_{ik}(t)$ of the matrices $A(t)$ and $B(t)$ are piecewise-continuous functions, bounded on $[0, T]$.

The control vector $u(t) = (u_1(t), \dots, u_r(t))$ is chosen from some set $U \subset E^r$. In what follows we shall assume that system (7)

and the set U satisfy the conditions under which, at time T , the attainability set

$$\Omega_T = \left\{ x(T) \mid x(T) = Y(T) \left(x_0 + \int_0^T Y^{-1}(t)B(t)u(t) dt \right), u(t) \in U \right\}$$

is bounded, closed, and strictly convex. Here $Y(t)$ is the fundamental matrix of system (7).

Let, as in item 4, the functions $f_i(x_1, \dots, x_i, \dots, x_n)$ ($i = 1, \dots, n$) determine the prices of the products.

Consider the problem of finding a control $u^*(t) = (u_1^*(t), \dots, u_r^*(t))$ such that the trajectory $x^*(t) = (x_1^*(t), \dots, x_n^*(t))$ of system (7) at time $t = T$ determines a point realizing

$$\max \left\{ \sum_{i=1}^n f_i(x^*(T))x_i \mid x \in \Omega_T \right\} = \sum_{i=1}^n f_i(x^*(T))x_i^*(T).$$

In other words, one seeks a control of the economy $u^*(t)$ that leads at time $t = T$ to the production of such quantities $x_i^*(t)$ ($i = 1, \dots, n$) of products that,

at prices $f_i(x^*(T))$ ($i = 1, \dots, n$), the vector $x^*(T)$ maximizes the value of the output produced.

For the dynamic version of A. Wald's model there is a statement analogous to theorem 2, from which, in particular, the existence of the vector $x^*(T)$ follows.

As already noted above, the algorithms of item 2 may be used to find the vector $x^*(T)$. In this case, at each step, the linear optimal-control problem (8) is solved, consisting in finding the maximum of a linear function on the set of endpoints of trajectories Ω_T .

Let us dwell in more detail on the first algorithm. As the initial approximation, choose a vector $x^0(T) \in \Omega_T$. Suppose that the approximations $x^1(T), \dots, x^k(T)$ have already been constructed.

- 1) Find a solution $\psi^k(t)$ of the system

$$d\psi/dt = -A^*(t)\psi, \quad \psi_j^k(T) = f_j(x^k(T)) \quad (j = 1, \dots, n).$$

- 2) Find the vector $u^k(t)$ from the condition that, at every time $t \in [0, T]$, the equality

$$(\psi^k(t), B(t)u^k(t)) = \max(\psi^k(t), B(t)u)$$

hold; that is, the vector $u^k(t)$ is found from L. S. Pontryagin's maximum principle.

- 3) Find the new approximation $x^{k+1}(T)$ by the formula

$$x^{k+1}(T) = (1 - \lambda_k)x^k(T) + \lambda_{kY}(T) \left(x_0 + \int_0^T Y^{-1}(t)B(t)u^k(t) dt \right),$$

where: a) $\lambda_k \rightarrow 0$, $0 < \lambda_k \leq 1$; b)

$$\sum_{k=0}^{\infty} \lambda_k = \infty.$$

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Note: Figure translations are in progress. See original paper for figures.

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