

VARIATIONAL PRINCIPLE AND CANONICAL VARIABLES IN MAGNE- TOHYDRODYNAMICS

V. E. Zakharov, E. A. Kuznetsov

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Abstract

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HYDROMECHANICS

V. E. ZAKHAROV, E. A. KUZNETSOV

VARIATIONAL PRINCIPLE AND CANONICAL VARIABLES IN MAGNETOHYDRODYNAMICS

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B. I. Davydov (¹) formulated a variational principle for the hydrodynamics of an ideal fluid and showed that the canonical variables are the pairs (λ, μ) and (ρ, Φ) , where

$$\mathbf{v} = \frac{\lambda}{\rho} \nabla \mu + \nabla \Phi.$$

In the present work the analogous problem is solved for the magnetohydrodynamics of an ideal fluid (MHD).

Let us consider the MHD equations:

$$\rho \left(\frac{\partial}{\partial t} \mathbf{v} + (\mathbf{v} \nabla) \mathbf{v} \right) = -\nabla p + \frac{1}{4\pi} [\text{rot } \mathbf{H}, \mathbf{H}]; \quad (1)$$

$$\partial \rho / \partial t + \text{div } \rho \mathbf{v} = 0; \quad (2)$$

$$\partial \mathbf{H} / \partial t = \text{rot}[\mathbf{v}, \mathbf{H}]; \quad (3)$$

$$\omega = \frac{\partial}{\partial \rho} (\rho \varepsilon) = \int dp / \rho.$$

Here ω is the enthalpy, and ε is the internal energy.

We make the change of variables:

$$\mathbf{v} = \frac{1}{\rho} [\mathbf{H}, \text{rot } \mathbf{S}] + \nabla \Phi. \quad (4)$$

For arbitrary $\rho, \mathbf{v}, \mathbf{H}$, from formula (4) one can reconstruct $\mathbf{S}$ and Φ , moreover nonuniquely, up to the addition to Φ of a scalar Φ_0 and to $\mathbf{S}$ of a vector $\mathbf{S}_0$ satisfying the equation

$$\frac{1}{\rho}[\mathbf{H}, \text{rot } \mathbf{S}_0] + \nabla \Phi_0 = 0.$$

After substituting formula (4) into equation (1), we obtain

$$\nabla \left\{ \frac{\partial \Phi}{\partial t} + (\mathbf{v} \nabla) \Phi - \frac{\mathbf{v}^2}{2} + \omega(\rho) \right\} + \left[\frac{\mathbf{H}}{\rho}, \text{rot} \left\{ \frac{\partial \mathbf{S}}{\partial t} - [\mathbf{v}, \text{rot } \mathbf{S}] + \frac{\mathbf{H}}{4\pi} \right\} \right] = 0. \quad (5)$$

Consider the system of equations

$$\frac{\partial \Phi}{\partial t} + (\mathbf{v} \nabla) \Phi - \frac{\mathbf{v}^2}{2} + \omega(\rho) = 0; \quad (6)$$

$$\frac{\partial \mathbf{S}}{\partial t} + \frac{\mathbf{H}}{4\pi} - [\mathbf{v}, \text{rot } \mathbf{S}] + \Delta \Psi = 0 \quad (7)$$

together with equations (2), (3). In equation (7), Ψ is an arbitrary function of the coordinates and time.

By virtue of formula (5), every solution of the system (2), (3), (6), (7) gives rise to some solution of the system of MHD equations (1)–(3). Under the assumption of uniqueness of the Cauchy solution for the system (1)–(3) and the system (2),

(3), (6), (7) the converse is also true: to any solution of the system (1)–(3) one can assign a certain class of solutions of the system (2), (3), (6), (7). For this it is sufficient, from the set of values of $\mathbf{v}, \mathbf{H}, \rho$ at some instant t_0 , to construct all possible sets of the quantities S, Φ satisfying formula (4), and to take them as the initial conditions for (2), (3), (6), (7).

Thus, the system of equations (2), (3), (6), (7) is equivalent to the system of MHD equations.

Let us choose Ψ in such a way that the relation

$$\frac{\partial}{\partial t} \text{div } \mathbf{S} = 0. \quad (8)$$

is satisfied. Obviously, this gives

$$\Psi = \frac{1}{\Delta} \text{div}[\mathbf{v}, \text{rot } \mathbf{S}] + \Psi_0,$$

where Ψ_0 is an arbitrary solution of Laplace's equation $\Delta\Psi_0 = 0$. In the particular case when, as $|\mathbf{r}| \rightarrow \infty$, $\mathbf{v} \rightarrow 0$, $\rho \rightarrow \rho_0$, $\mathbf{H} \rightarrow \mathbf{H}_0$, it is convenient to choose the quantity Ψ_0 so that $\partial\mathbf{S}/\partial t \rightarrow 0$ as $|\mathbf{r}| \rightarrow \infty$. Obviously, then

$$\Psi_0 = -\frac{1}{4\pi}(\mathbf{H}_0, \mathbf{r}).$$

Consider the functional

$$E = \int \left\{ \rho \frac{\mathbf{v}^2}{2} + \rho\varepsilon + \frac{\mathbf{H}^2}{8\pi} - \Psi \operatorname{div} \mathbf{H} \right\} d\mathbf{r}. \quad (9)$$

By direct variation of this functional we find that the equations

$$\frac{\partial \rho}{\partial t} = \frac{\delta E}{\delta \Phi}; \quad \frac{\partial \Phi}{\partial t} = -\frac{\delta E}{\delta \rho}; \quad \frac{\partial \mathbf{H}}{\partial t} = \frac{\delta E}{\delta \mathbf{S}}; \quad \frac{\partial \mathbf{S}}{\partial t} = -\frac{\delta E}{\partial \mathbf{H}} \quad (10)$$

coincide with equations (2), (6), (3), (8). Thus, the variables (ρ, Φ) and $(\mathbf{H}, \mathbf{S})$ are pairs of canonically conjugate quantities, and the functional E is the Hamiltonian.

Introduce the Lagrangian function

$$\mathcal{L} = \int \left\{ \left(\mathbf{S}, \frac{\partial \mathbf{H}}{\partial t} \right) + \Phi \frac{\partial \rho}{\partial t} \right\} d\mathbf{r} - E$$

and construct the action functional

$$S = \int \mathcal{L} dt,$$

whose minimization gives a variational principle for the MHD equations. Note that for real motion $\operatorname{div} \mathbf{H} = 0$, and the functional E coincides with the total energy of the fluid.

In the case of an incompressible fluid the quantity Φ is eliminated with the aid of the continuity equation

$$\Delta\Phi = -\operatorname{div} \frac{1}{\rho_0}[\mathbf{H}, \operatorname{rot} \mathbf{S}].$$

The canonical variables are $(\mathbf{H}, \mathbf{S})$, and the Hamiltonian has the form

$$E = \int \left\{ \frac{\rho_0 \mathbf{v}^2}{2} + \frac{\mathbf{H}^2}{8\pi} - \Psi \operatorname{div} \mathbf{H} \right\} d\mathbf{r}.$$

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REFERENCES

1. B. I. Davydov, DAN, **69**, No. 2, 165 (1949).

Note: Figure translations are in progress. See original paper for figures.

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