

# STUDIES ON MULTIDIMENSIONAL LIMIT THEOREMS OF PROBABILITY THEORY

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**Abstract**

**Full Text**

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**MATHEMATICS**

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## STUDIES ON MULTIDIMENSIONAL LIMIT THEOREMS OF PROBABILITY THEORY

*(Presented by Academician Yu. V. Linnik on 16 IX 1969)*

We consider the asymptotic behavior of the density  $p_n(\mathbf{x})$  and of the probability function  $P_n(A)$  of the sum

$$S_n = \frac{1}{\sqrt{n}} \sum_{j=1}^n (\xi_j - M\xi_j)$$

of independent identically distributed  $k$ -dimensional random vectors  $\xi_j = (\xi_{1j}, \xi_{2j}, \dots, \xi_{kj})$ ,  $j = 1, 2, \dots, n$ , with nondegenerate covariance matrix  $V$ , where  $M\xi_j$  is the vector of mathematical expectations.

The present paper is a continuation of the author's investigations <sup>(1,2)</sup> and refines a number of known <sup>(3-12)</sup> multidimensional limit theorems.

1. In order to reveal the structure of the remainder term in the asymptotic expansions for  $P_n(A)$  and  $p_n(\mathbf{x})$ , an upper estimate has been obtained for the modulus of the characteristic function  $f(\mathbf{t})$  of the random vector  $\xi_1$ , whose distribution function has a density  $q(\mathbf{x})$  bounded by a constant  $C$ . It should be noted that S. M. Sadikova obtained <sup>(6)</sup> sufficient conditions (certain restrictions on the density  $q(\mathbf{x})$ ) for the power decrease of the modulus of the characteristic function  $f(\mathbf{t})$  as  $\|\mathbf{t}\| \rightarrow \infty$ , where  $\|\mathbf{t}\|$  is the length of the vector  $\mathbf{t} \in R^k$ . We have obtained the following lemma, from which for  $k = 1$  the known <sup>(13)</sup> result for one-dimensional characteristic functions follows.

**Lemma 1.** For all  $\mathbf{t} \in R^k$  the inequality holds

$$|f(\mathbf{t})| \leq \exp \left\{ -\frac{\pi^2}{27} \frac{(\mathbf{t}V\mathbf{t}')((k-1)!!)^2}{C^2|V|(8\pi)^k k^{k-1} (2\pi + \sqrt{k}\sqrt{\mathbf{t}V\mathbf{t}'})^2} \right\}. \quad (1)$$

Here  $\mathbf{t}V\mathbf{t}'$  is the quadratic form corresponding to the matrix  $V$ ;  $|V|$  is the determinant of the matrix  $V$ .

From formula (10) of paper <sup>(2)</sup> and inequality (1) it follows that

**Theorem 1.** If the distribution function of the random vector  $\xi_1$  has a density bounded by a constant  $C$ , and finite moments of order  $s$  ( $s \geq 3$ ), then for all  $\mathbf{x} \in R^k$

$$\left| p_n(\mathbf{x}) - \sum_{j=0}^{s-3} \left( \frac{1}{\sqrt{n}} \right)^j P_j(-\varphi)(\mathbf{x}) \right| \leq \frac{2^{s-1} \beta_s \Gamma((k+s)/2)}{\pi^{k/2} \sqrt{|V|} \Gamma(k/2) n^{(s-2)/2}} + \\ + C n^{k/2} \exp \left\{ -\frac{\pi^2}{27} \frac{(n-2)((k-1)!!)^2}{C^2 \sqrt{|V|} (8\pi)^k k^{k-1} (16\pi \beta_s^{1/(s-2)} + \sqrt{k})^2} \right\}.$$

Here  $P_j(-\varphi)(\mathbf{x})$  are known functions of many variables,

$$\beta_s = \sup_{\|\mathbf{t}\|=1} \frac{M|(\xi_1 - M\xi_1, \mathbf{t})|^s}{(M(\xi_1 - M\xi_1, \mathbf{t})^2)^{s/2}}, \quad \Gamma(\alpha) = \int_0^\infty u^{\alpha-1} e^{-u} du,$$

$\|\mathbf{t}\|$  is the modulus of the vector  $\mathbf{t} \in R^k$ , and  $(\xi_1, \mathbf{t})$  is the scalar product.

In the proof of Theorem 1 we shall use the following lemma.

**Lemma 2.** If  $\xi_1$  has finite moments of order  $s$  ( $s \geq 3$ ), then, for  $(M|(\xi_1 - M\xi_1, t)|^s / tVt')^{1/(s-2)} \leq \sqrt{n}/8$ ,

$$\left| f^n \left( \frac{t}{\sqrt{n}} \right) - \exp \left( \frac{tVt'}{2} \right) \left( 1 + \sum_{j=1}^{s-3} \left( \frac{1}{\sqrt{n}} \right)^j P_j(it) \right) \right| \leq \frac{2^{3s/2} M|(\xi_1 - M\xi_1, t)|^s}{n^{(s-2)/2}} \exp \left( -\frac{tVt'}{4} \right);$$

$P_j(it)$  are known polynomials.

2. Let us denote by  $\mathfrak{R}$  and  $\mathfrak{A}$ , respectively, the classes of Borel and convex sets  $A$  in  $R^k$ ;  $\Phi(\mathbf{x})$  is the  $k$ -dimensional normal distribution with parameters  $(0, V)$ , where  $0$  is the zero vector in  $R^k$ ;

$$G_{s-2}(A) = \sum_{j=0}^{s-2} \left( \frac{1}{\sqrt{n}} \right)^j \int_A P_j(-\varphi)(\mathbf{y}) d\mathbf{y};$$

$C_1, C_2, \dots$  are positive constants not depending on the set  $A$  or on the number of summands  $n$  in the sum  $S_n$ ;  $\Phi_T(\mathbf{x})$  is the  $k$ -dimensional distribution function with characteristic function  $\exp\{-\|\mathbf{t}\|^2/T\}$ ,  $T > 0$ ;  $H_T(\mathbf{x})$  is the distribution function with characteristic function  $h(t/T)$ , where

$$h(\mathbf{t}) = \int_{R^k} e^{i(\mathbf{t}, \mathbf{x})} \prod_{j=1}^k \frac{96}{\pi} \left( \frac{\sin x_j/4}{x_j} \right)^4 d\mathbf{x};$$

$(\mathbf{t}, \mathbf{x})$  is the scalar product of the vectors  $\mathbf{t}, \mathbf{x} \in R^k$ .

By the method of truncating random vectors (see (2)) the following theorems are proved.

**Theorem 2.** If  $\xi_1$  has finite moments of order  $s$  ( $s \geq 3$ ) and Cramér's condition is satisfied,

$$\lim_{\|t\| \rightarrow \infty} |f(t)| < 1, \quad (c)$$

then, uniformly in  $A \in \mathfrak{A}$ , the relation

$$|(P_n - G_{s-2}) * \Phi_T(A)| \leq d_1(n)/n^{(s-2)/2}$$

holds for  $T = (C_1 n)^{(s-1)/2}$ . Here and below  $\lim d_i(n) = 0$ ,  $i = 1, 2, \dots$ ;  $*$  denotes convolution.

**Theorem 3.** If  $\xi_1$  has finite moments of order  $2 + \delta$  ( $0 < \delta \leq 1$ ), then, uniformly in  $A \in \mathfrak{A}$ ,

$$|(P_n - \Phi) * H_T(A)| \leq C_2/n^{\delta/2}$$

for  $T = (C_3 n)^{\delta/2}$ .

**Theorem 4.** Suppose that the distribution function of the random vector  $\xi_1$  is non-lattice, i.e.  $|f(t)| = 1$  only at the single finite point  $t = 0$ , and has finite moments of third order; then, uniformly in  $A \in \mathfrak{A}$ ,

$$|(P_n - G_1) * H_T(A)| \leq d_2(n)/\sqrt{n}$$

for  $T = \lambda(n)\sqrt{n}$ , where  $\lim_{n \rightarrow \infty} \lambda(n) = \infty$ .

With the aid of the remarkable so-called "inversion formula" of B. Bahr (10), under widely accepted natural conditions, for broad classes of sets  $A$  there are obtained (Theorems 2'-4') asymptotic expansions for the probability function  $P_n(A)$ . The estimates obtained for the remainder terms are optimal with respect to  $n$  and sharper than the known ones (see (4-6, 10, 12)).

**Theorem 2'.** Under the conditions of Theorem 2 we have

$$|P_n(A) - G_{s-2}(A)| \leq \frac{d_1(n)}{n^{(s-2)/2}} + 2\alpha (A_1(C_1 n)^{-(s-1)/2})$$

for  $A \in \mathfrak{A}$  and

$$\sup_{A \in \mathfrak{A}} |P_n(A) - G_{s-2}(A)| = o(n^{-(s-2)/2}). \quad (2)$$

**Theorem 3'.** Under the conditions of Theorem 3 we have

$$|P_n(A) - \Phi(A)| \leq C_2 n^{-\delta/2} + 2\alpha(A_1(C_3 n)^{-\delta/2})$$

for  $A \in \mathfrak{A}$ , and

$$\sup_{A \in \mathfrak{A}} |P_n(A) - \Phi(A)| \leq C_4 n^{-\delta/2}.$$

**Theorem 4'.** Under the conditions of Theorem 4 we have

$$|P_n(A) - G_1(A)| \leq d_2(n)/\sqrt{n} + 2\alpha(A, 1/\lambda(n)\sqrt{n})$$

for  $A \in \mathfrak{A}$ , and

$$\sup_{A \in \mathfrak{A}} |P_n(A) - G_1(A)| = o(1/\sqrt{n}).$$

Here

$$\alpha(A, \varepsilon) = \sup_{y \in R^k} \int_{(A)_\varepsilon} |dQ(\mathbf{x} + \mathbf{y})|,$$

where  $(A)_\varepsilon$  is the  $\varepsilon$ -neighborhood of the set  $A$ .

In Theorems 2', 3', and 4', respectively,  $Q(\mathbf{x}) = G_{s-2}(\mathbf{x})$ ,  $\Phi(\mathbf{x})$ , and  $G_1(\mathbf{x})$ . Theorems 2'–4' are new also for one-dimensional probability functions  $P_n(A)$ .

It should be noted that for the class of  $k$ -dimensional “intervals” relation (2) was obtained by different methods by the author in (2) and by B. Bahr in (10).

3. Finally, by the method of truncating random vectors from the local limit theorem we obtain the following asymptotic expansion, which includes the known result of R. R. Rao (5).

**Theorem 5.** Let the random vector  $\xi_1$  take values from the  $k$ -dimensional lattice

$$\{\mathbf{a} + \mathbf{v}H : \mathbf{v} = (v_1, v_2, \dots, v_k), v_1, v_2, \dots, v_k = 0, \pm 1, \pm 2, \dots\}$$

with maximal “span of the distribution”  $h = |\det H|$ , and have finite moments of order  $s$  ( $s \geq 3$ ); then, uniformly in  $A \in \mathfrak{A}$ ,

$$P_n(A) = \int d \prod_{l=1}^k \left\{ \sum_{\nu=0}^{s-2} \frac{(-1)^\nu}{\nu!} \left( \frac{h}{\sqrt{n} A_{ll}} \right)^\nu B_\nu \left( \left( \frac{\sqrt{n} \mathbf{y} + n M \xi_1 - n \mathbf{a}}{n}, \mathbf{q}_l \right) \right) \times \frac{\partial^\nu}{\partial y_l^\nu} \right\} G(\mathbf{y}) + o(n^{-(s-2)/2}). \quad (3)$$

Here the vector  $\mathbf{q}_l$  has coordinates  $A_{lj}$ ,  $j = 1, 2, \dots, k$ ;  $A_{lj}$  is the algebraic cofactor of the element with indices  $l, j$  in the determinant of the transposed matrix  $H'$ ;  $B_0(x) \equiv 1$ ,

$$-\frac{B_\nu(x)}{\nu!} = \sum_{m=-\infty}^{\infty} \frac{e^{2\pi i m x}}{(2\pi i m)^\nu}, \quad \nu = 2, 3, \dots$$

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