

ISOMETRIC EMBEDDINGS AND IMMERSIONS

M. L. Gromov

1970

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-197001.36592>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 513.836

MATHEMATICS

M. L. GROMOV

ISOMETRIC EMBEDDINGS AND IMMERSIONS

(Presented by Academician L. S. Pontryagin on 16 XII 1969)

I. Introduction. In the present note theorems are formulated that describe the structure of the space of isometric immersions of one pseudo-Riemannian (in particular, Riemannian) manifold into another. These theorems generalize and refine the results of Nash and Kuiper on isometric C^1 -immersions, Nash's theorems on isometric embeddings of the classes C^∞, C^a (analytic), the local theorem of Janet–Cartan–Burstin, and also the results of paper ⁽⁵⁾, which, in particular, contains a survey of the question and the literature. From the theorems of Sec. III of the present work there follow the immersion theorems of Smale–Hirsch and their generalizations (see ⁽²⁾). The propositions of Sec. VII are adjacent to paper ⁽⁴⁾.

II. Auxiliary notions. If η, θ are real vector bundles over A with fibers η_a, θ_a , $a \in A$, then a homomorphism (monomorphism) $\eta \rightarrow \theta$ is called a continuous family of linear (linear one-to-one) mappings $\eta_a \rightarrow \theta_a$. If ξ is a bundle over B and $f : A \rightarrow B$ is a continuous mapping, then the bundle induced over A is denoted by $f^*(\xi)$. A morphism (injection) of the bundle η over A into the bundle ξ over B is a pair (f, φ) , where $f : A \rightarrow B$ is a continuous mapping and $\varphi : \eta \rightarrow f^*(\xi)$ is a homomorphism (monomorphism). For a morphism $\psi = (f, \varphi)$ we put $|\psi| = \varphi$. A form g in the bundle η over A is a continuous family of quadratic forms g_a , $a \in A$, in the fibers η_a . If the form g is nondegenerate, i.e. is reducible in each fiber to the form

$$\sum_{i=1}^k x_i^2 - \sum_{i=k+1}^n x_i^2$$

with $n = \dim \eta$, then it is called a metric; by $\text{pos } g$ we denote the rank k of the positive part, and by $\text{neg } g$ the rank $n - k$ of the negative part of the metric g . A nondegenerate form g with $\text{neg } g = 0$ is called positive.

In what follows W denotes a manifold with tangent bundle ω and metric h in ω (i.e. W is a pseudo-Riemannian manifold with metric h). If $f : A \rightarrow W$ is a continuous mapping, then by $f^*(h)$ we denote the metric induced in the bundle $f^*(\omega)$; for a homomorphism φ of the bundle η over A into the bundle $f^*(\omega)$, by

$g(\varphi)$ we denote the form in η induced by it (from the metric $f^*(h)$); and for a morphism $\psi = (f, \varphi) : \eta \rightarrow \omega$ we put $g(\psi) = g(\varphi)$.

In the sequel V denotes a manifold with a distribution α (a distribution α on V is a subbundle α of the tangent bundle $\tau(V)$, i.e. a field of tangent planes on V) and a metric g in α . A homomorphism φ of the bundle α into the bundle $f^*(\omega)$ induced by a continuous mapping $f : V \rightarrow W$ is called isometric if $g(\varphi) = g$. Analogously, a morphism $\psi : \alpha \rightarrow \omega$ is called isometric if $g(\psi) = g$. If $f : V \rightarrow W$ is a smooth mapping, then by $\delta_f : \alpha \rightarrow \omega$ we denote the restriction to α of the differential $d_f : \tau(V) \rightarrow \omega$. A mapping $f : V \rightarrow W$ of class C^r , $r = 1, 2, \dots, \infty, a$, will be called a C^r -isometry if the morphism δ_f is isometric. A C^1 -mapping $f : V \rightarrow W$ will be called short if the form $g - g(\delta_f)$ is positive.

In this work the manifolds, bundles, and metrics are assumed to be C^∞ -smooth, and the notion of approximation is used with respect to the space $C^r(V, W)$, $0 \leq r \leq \infty$, endowed with the fine C^r -topology (see ⁽⁶⁾).

A C^1 -isometry $V \rightarrow W$ will be called **non-loose** if it admits a C^1 -approximation by short mappings $V \rightarrow W$. For example, if $W = R^q$, and the manifold V is compact, or if $\text{neg } h > \text{neg } g$, then every isometry $V \rightarrow W$ is non-loose, while the standard isometry $R^{q-k} \rightarrow R^q$ is not non-loose.

III. C^1 -isometries.

Theorem 1. *Let $f : V \rightarrow W$ be a smooth mapping and suppose that one of the following two conditions is satisfied:*

- a) $\text{pos } h > \text{pos } g$, $\text{neg } h > \text{neg } g$,
- b) $\text{pos } h > \text{pos } g$, $\text{neg } g = 0$ and f is a short mapping.

Then, for the existence of a C^0 -approximation of the mapping f by non-loose C^1 -isometries $V \rightarrow W$, it is necessary and sufficient that there exist an isometric homomorphism $\alpha \rightarrow f^*(\omega)$.

Corollary. A. *If the manifold V is contractible, $\text{pos } h > \text{pos } g$, $\text{neg } h > \text{neg } g$, then every continuous mapping $V \rightarrow W$ is approximated by C^1 -isometries.*

A manifold M is called a π -manifold if the tangent bundle of the manifold $M \times R^1$ is trivial; an example of a π -manifold is the n -dimensional sphere.

B. *A short C^1 -mapping of an n -dimensional Riemannian π -manifold into R^{n+1} admits a C^0 -approximation by C^1 -isometries. (We note that a short mapping is not assumed to be an immersion.)*

C. *For any linearly independent vector fields $X_1, \dots, X_n$ on a Riemannian manifold M , there exist C^1 -functions $f_1, \dots, f_{n+1} : M \rightarrow R^1$ such that*

$$\sum_{i=1}^{n+1} (X_k f_i)(X_l f_i) = (X_k, X_l), \quad 1 \leq k, l \leq n.$$

(Here Xf denotes the derivative of the function f along X (see (1)), and (X, Y) denotes the scalar product of the fields X and Y .)

Theorem 2. *The mapping from the space of C^1 -isometries $V \rightarrow W$ to the space of isometric morphisms $\alpha \rightarrow \omega$, which assigns to an isometry f the morphism δ_f , is a weak homotopy equivalence (i.e., induces an isomorphism of homotopy groups) in the following two cases:*

- a) $\text{pos } h > \text{pos } g, \text{ neg } h > \text{neg } g$;
- b) $W = R^q$ with $q > \text{pos } g$.

Corollary. *Two homotopic C^1 -isometries $V \rightarrow W$ with $\text{pos } h > \text{pos } g + \dim V$, $\text{neg } h > \text{neg } g + \dim V$ can be joined by a path in the space of C^1 -isometries $V \rightarrow W$.*

IV. **C^1 -embeddings.** Two monomorphisms $\eta \rightarrow \theta$ are called homotopic if they are connected by a continuous family of monomorphisms $\eta \rightarrow \theta$.

Theorem 3. *Suppose that $\alpha = \tau(V)$ (i.e., V is a pseudo-Riemannian manifold with metric g). Let $f : V \rightarrow W$ be a smooth embedding and suppose that one of the following two conditions is satisfied:*

- a) $\text{pos } h > \text{pos } g, \text{ neg } h > \text{neg } g$,
- b) $\text{pos } h > \text{pos } g, \text{ neg } g = 0$ and f is a short mapping.

Then, for the existence of a C^0 -approximation of the mapping f by non-loose isometric C^1 -embeddings $V \rightarrow W$, diffeotopic to the embedding f , it is necessary and sufficient that there exist an isometric monomorphism $\alpha \rightarrow f^*(\omega)$ homotopic to the monomorphism $|\delta_f| = |d_f| : \alpha \rightarrow f^*(\omega)$.

Corollary. *If V is compact, then every continuous mapping $V \rightarrow W$ with $\text{pos } h > \text{pos } g + \dim V$, $\text{neg } h \geq \text{neg } g + \dim V$ is approximated by isometric C^1 -embeddings.*

V. **Free immersions.** For a bundle η over A , by η^2 we denote its symmetric square. By $X \circ Y \in (\eta^2)_a$, $a \in A$, we denote the vector corresponding to the pair of vectors $X, Y \in \eta_a$. (The symmetric square η^2 is the bundle associated with η , whose fiber $(\eta^2)_a$, $a \in A$, is the symmetric product $\eta_a \circ \eta_a$; see (2).)

Further, the distribution α on V is assumed to be involutive. (A distribution α is called involutive (see (4)) if the Poisson bracket of two vector fields belonging to α also belongs to α .)

Consider a C^2 mapping $f : V \rightarrow W$. Suppose that the form $g(|\delta_f|) = g(\delta_f)$ is nondegenerate, and denote by $P : f^*(\omega) \rightarrow f^*(\omega)$ the projection (orthogonal with respect to the metric $f^*(h)$) onto the orthogonal complement of the image of the bundle α under the homomorphism $|\delta_f| : \alpha \rightarrow f^*(\omega)$. Define the homomorphism $|\delta_f|^2 : \alpha^2 \rightarrow f^*(\omega)$ by the following condition. If $X, Y \in \alpha_v$, $v \in V$, $\bar{Y}$ is a smooth section of the bundle α extending the vector Y , and Y_1 is the

image of the section $\bar{Y}$ under the homomorphism $|\delta_f|$, then

$$|\delta_f|^2(X \circ Y) = P\nabla_X Y_1,$$

where by $\nabla_X Y_1 \in (f^*(\omega))_v$ is denoted the covariant derivative along X of the section Y^0 of the bundle $f^*(\omega)$ with the connection induced from the pseudo-Riemannian connection in ω . A C^2 mapping $f : V \rightarrow W$ will be called α -free if the form $g(\delta_f)$ induced in α is nondegenerate, and the form $g(|\delta_f|^2)$ induced in α^2 is positive. If $\alpha = \tau(V)$, then an α -free mapping is called free. For example, an immersion $R^1 \rightarrow W$ is free if it induces in R^1 a nondegenerate form, its geodesic curvature does not vanish, and on the principal normal the form h is positive.

Theorem 4. *If the manifolds V, W , the distribution α , and the metrics g, h are analytic, then an α -free C^∞ -isometry $V \rightarrow W$ admits a C^∞ -approximation by analytic isometries $V \rightarrow W$.*

Next put $n = \dim \alpha$, $s = s_n = \dim \alpha^2 = n(n+1)/2$.

Theorem 5. *In order that a given C^1 -isometry $f : V \rightarrow W$ admit a C^1 -approximation by α -free C^∞ -isometries $V \rightarrow W$ for*

$$\text{pos } h \geq s + \text{pos } g + 2n + 5,$$

it is necessary and sufficient that there exist a monomorphism $\varphi : \alpha^2 \rightarrow f^*(\omega)$ whose image is orthogonal (with respect to the metric $f^*(h)$) to the image of the homomorphism $|\delta_f| : \alpha \rightarrow f^*(\omega)$ and for which the form $g(\varphi)$ is positive.

Combining Theorems 3, 4, 5 leads to the corollaries:

Corollary. A. *For*

$$\text{neg } h \geq \text{neg } g + \dim V, \quad \text{pos } h \geq s + \dim V + 2n + 5$$

every continuous mapping $V \rightarrow W$ admits a C^0 -approximation by α -free C^∞ -isometries $V \rightarrow W$.

B. *An n -dimensional Riemannian manifold of class C^i , $i = \infty, a$, has an isometric C^i -embedding in R^q with $q = s + 3n + 5$, and for a complete manifold the embedding may be chosen without a limit set; if the manifold is compact and analytic, then it admits an isometric C^a -embedding in R^q with $q = s + 3n + 4$.*

Call a morphism $\varphi : \alpha \oplus \alpha^2 \rightarrow \omega$ (where $\alpha \oplus \alpha^2$ denotes the Whitney sum of the bundles α and α^2) semi-isometric if the form $g(\varphi)$ induced by it in $\alpha \oplus \alpha^2$ is nondegenerate, coincides on the subbundle α with g , and

$$\text{neg } g(\varphi) = \text{neg } g.$$

Theorem 6. *The mapping of the space of α -free C^∞ -isometries $V \rightarrow W$ into the space of semi-isometric morphisms $\alpha \oplus \alpha^2 \rightarrow \omega$, which assigns to an isometry f the morphism*

$$(f, |\delta_f| \oplus |\delta_f|^2) : \alpha \oplus \alpha^2 \rightarrow \omega$$

(where $|\delta_f| \oplus |\delta_f|^2 : \alpha \oplus \alpha^2 \rightarrow f^*(\omega)$ denotes the direct sum of the homomorphisms $|\delta_f|$ and $|\delta_f|^2$), is a weak homotopy equivalence in the following two cases:

- a) $\text{pos } h \geq s + \text{pos } g + 2n + 5, \quad \text{neg } h > \text{neg } g;$
- b) $W = R^q, \quad q \geq s + 3n + 5.$

Corollary. Two free isometric immersions of an n -dimensional Riemannian manifold in R^q with

$$q \geq s + 3n + 5$$

are connected by a C^∞ bending.

VI. Riemannian manifolds. The aim of what follows is to remove, in a number of cases, the dimensional restrictions of the preceding section.

Theorem 7. Let V be an n -dimensional Riemannian manifold, V_0 its connected closed submanifold, and W a contractible Riemannian manifold. Denote by β the restriction to V_0 of the tangent bundle $\tau(V)$, and by ν the normal bundle of the submanifold V_0 .

- a) If ν is a trivial one-dimensional bundle, and the manifold V_0 , as a Riemannian manifold with the metric induced from V , is reducible, or if $\dim \nu \geq 2$, then, in order that some neighborhood $U \subset V$ containing the submanifold V_0 have a free isometric embedding in W , it is necessary and sufficient that there exist an injection $\beta \oplus \beta^2 \rightarrow \tau(W)$. (For a π -manifold V an injection $\beta \oplus \beta^2 \rightarrow \tau(W)$ exists when $\dim W \geq s + n$.)
- b) If the manifolds V , W and the submanifold V_0 are analytic, the bundle β is trivial, and the bundle ν has a nonvanishing section and $\dim \nu \geq 3$, then for $\dim W \geq s = n(n+1)/2$ some neighborhood $U \subset V$ containing the submanifold V_0 has an analytic isometric embedding in W .

Theorem 8. Let V_0 be a compact $(n-1)$ -dimensional Riemannian π -manifold, and let $f_1, f_2 : V_0 \rightarrow R^q$ be free isometric C^∞ -immersions (C^∞ -embeddings). If $q \geq s + 2n + 2$ ($s = n(n+1)/2$), then for some $l > 0$ there exists a free isometric C^∞ -immersion (C^∞ -embedding) f of the cylinder $V_0 \times [0, l]$ into R^q , such that $f|_{V_0 \times 0} = f_1, f|_{V_0 \times l} = f_2$.

VII. Topological propositions. A. Let M be a smooth open π -manifold, W a Riemannian manifold, $f : M \rightarrow W$ a continuous map, and k a real number. For the existence of a free C^∞ -immersion $M \rightarrow W$ homotopic to the map f and inducing on M a metric of constant curvature k , it is necessary and sufficient that there exist a monomorphism $\tau(M) \oplus (\tau(M))^2 \rightarrow f^*(\tau(W))$.

B. Let M be a smooth n -dimensional π -manifold and let $f, g : M \rightarrow R^q$ be continuous maps. If $q > s/2 + n$ ($s = n(n+1)/2$), then f, g admit C^0 -approximation by C^∞ -immersions $f_1, g_1 : M \rightarrow R^q$ inducing on M one and the same metric.

I thank Prof. V. A. Rokhlin, who attracted me to this subject and to the seminar on isometric embeddings organized by him. I express my gratitude to the leader and participants of this seminar, and also to V. L. Eidlin for discussions.

Leningrad State University
named after A. A. Zhdanov

Received
3 XII 1969

CITED LITERATURE

1. R. Bishop, R. Crittenden, *Geometry of Manifolds*, Moscow, 1967.
2. J. M. Boardman, in: *Singularities of Differentiable Mappings*, Moscow, 1968.
3. M. L. Gromov, *Dokl. Akad. Nauk SSSR* **182**, No. 2, 255 (1968).
4. M. L. Gromov, *Izv. Akad. Nauk SSSR, Ser. Mat.* **33**, issue 4, 707 (1969).
5. M. L. Gromov, V. A. Rokhlin, *Uspekhi Mat. Nauk* **25**, issue 2 (1970).
6. J. Moser, in: *Singularities of Differentiable Mappings*, Moscow, 1968.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.