

# ILL-POSED PROBLEMS AND GEOMETRIES OF BANACH SPACES

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1970

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**Abstract**

**Full Text**

UDC 517.948

*MATHEMATICS*

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## ILL-POSED PROBLEMS AND GEOMETRIES OF BANACH SPACES

*(Presented by Academician A. N. Tikhonov on 8 I 1970)*

**1. Statement of the problem.** Let  $X$  be a reflexive space;  $Y$  a separable locally convex space, and  $A$  a linear continuous operator mapping  $X$  into  $Y$ , such that there exists an inverse operator  $A^{-1}$ , which, generally speaking, is not continuous. Consider the operator equation of the first kind

$$Ax = y_0, \quad x \in X, \quad y_0 \in R(A) \subset Y. \quad (1)$$

Suppose that  $y_0$  is not known to us, and instead of it there is given a basis of a filter of neighborhoods  $\{V_\delta\}$  of the point  $y_0$ , where each  $V_\delta$  is a convex closed set:  $V_\delta = \overline{\text{co}}(V_\delta)$ .

It is required, from the given  $V_\delta$ , to find an approximate solution  $x_\delta$  of equation (1) such that  $x_\delta \rightarrow x_0$ , where  $x_0$  is the exact solution of equation (1), i.e.  $x_0 = A^{-1}y_0$ .

**2. The residual method.** The idea of a stable solution of an operator equation of the first kind

$$\bar{A}[x] = u_0 \quad (x \in X, \quad u_0 \in U), \quad (2)$$

where  $\bar{A}[x]$  is a continuous operator from  $X$  into  $U$ , satisfying the uniqueness condition:  $\bar{A}[x_1] \neq \bar{A}[x_2]$  if  $x_1 \neq x_2$ ;  $X, U$  are linear normed spaces, was first put forward by A. N. Tikhonov <sup>(1-3)</sup>. Its basis was the additional assumption that the exact solution  $x_0$ :  $\bar{A}[x_0] = u_0$  of equation (2) belongs to some class of well-posedness (stabilization)  $m, m \subset X$ .

In <sup>(3)</sup>, for example, for solving equation (2) the idea of a compact embedding is developed. This idea may be interpreted, as was done by V. K. Ivanov in <sup>(5)</sup> for equation (2), as follows: it was assumed that there is a Hilbert space  $Z$ , which is mapped into  $X$  by means of a linear completely continuous operator  $B$ . It was further assumed that the exact solution  $x_0$  of equation (2) belongs to the range  $R(B)$  of the operator  $B$  in the space  $X$ :  $x_0 \in R(B)$ . The set  $R(B)$ ,

$R(B) \subset X$ , in this case plays the role of the stabilization class in the space  $X$ , and, generally speaking,  $R(B) \neq X$ . The residual method for solving equation (2) (see <sup>(5)</sup>) consisted in finding an element  $z_\delta$  with minimal norm on the set  $\Omega_\delta : \Omega_\delta = \{z : \|Cz - u_\delta\| \leq \delta\}$ , where  $C = \bar{A}B$  is a continuous operator acting from  $Z$  into  $U$ ,  $u_\delta \in U$ ,  $\|u_0 - u_\delta\| \leq \delta$ .

Let us note that if the operator  $\bar{A}$ , in addition, is linear, then for a stable solution of equation (2) it is sufficient to have merely a continuous embedding, i.e. the embedding operator  $B$ , mapping  $Z$  into  $X$ , may be assumed linear and continuous (see <sup>(13,14)</sup>). In connection with the idea of a continuous embedding, the case  $Z = X$ , and  $B$  the identity operator on  $Z$ , is of interest. In this case the stabilization class  $R(B)$  is maximal in the space  $X$ , or, in other words,  $R(B) = X$ . This circumstance is very convenient, since the necessity of checking whether the exact solution  $x_0$  of equation (2) belongs to the stabilization class  $R(B)$  is eliminated.

In the work of V. A. Morozov, for example (see <sup>(13)</sup>), the space  $X$  is assumed to be Hilbert, and the stabilization class to coincide with the whole  $X$ ,

and  $\bar{A}$  is a linear continuous operator mapping  $X$  into a linear normed space  $U$ , and under these assumptions a stable solution of equation (2) is given by the method of A. N. Tikhonov. However, even the assumption that the spaces  $X$  are Hilbert spaces is inessential (see <sup>(14, 16)</sup>). It is enough that  $X$  satisfy the Efimov-Stechkin condition (see <sup>(17)</sup>). An example of an Efimov-Stechkin space may be, for instance, the space  $L_p$  ( $p > 1$ ).

A natural question arises: in which spaces may one suppose the class of stabilization  $R(B)$  to coincide with the whole space  $X$ ? This note is devoted to the solution of this question. Therefore, in what follows we shall consider only the case  $Z = X$  and  $B$  is the identity operator on  $X$ . Then the residual method, as applied to the solution of the problem posed in Sec. 1, consists in finding a point of minimum of the functional

$$K[x] = \|x\| \quad (3)$$

under the condition

$$Ax \in V_\delta. \quad (4)$$

Since  $X$  is reflexive, the point of minimum exists (see <sup>(10)</sup>). This point is unique if  $X$  is strictly convex and reflexive (see <sup>(10)</sup>). Therefore, in what follows we shall always assume  $X$  to be strictly convex and reflexive. The points of minimum will be called solutions of the problem posed in Sec. 1, obtained by the residual method, and will be denoted by  $x_\delta$ . In what follows, the problem posed in Sec. 1 will be denoted by  $\theta$ :  $\theta = \theta[X, Y, x_0, A, \{V_\delta[Ax_0]\}]$ , where  $X \in \mathfrak{X}$ ,  $\mathfrak{X}$  is the class of reflexive strictly convex spaces;  $Y \in \mathfrak{Y}$ ,  $\mathfrak{Y}$  is the class of separable locally convex spaces;  $x_0 \in X$ ;  $A \in \mathcal{L}(X, Y)$ ,  $\mathcal{L}(X, Y)$  is the space of linear

continuous one-to-one operators acting from  $X$  into  $Y$ ;  $\{V_\delta[Ax_0]\} \in \mathfrak{B}$ ,  $\mathfrak{B}$  is the totality of all such bases of the filter of neighborhoods of the point  $Ax_0 \in Y$ , each of which consists only of convex closed sets of the space  $Y$ . By  $\Theta_X$  we shall denote the set of problems  $\theta$  for which  $X$ —a reflexive strictly convex space—is fixed, whereas the remaining components of the problem  $\theta$  range over the corresponding domains of definition.

### 3. Basic definitions.

**Definition 1.** The residual method will be called **stable for the problem  $\theta$**  if the solutions  $x_\delta$  of this problem, obtained by the residual method, converge strongly to the exact solution  $x_0$  of equation (1).

**Remark.** In this definition all components of the problem  $\theta$  are regarded as fixed.

**Definition 2.** The residual method will be called **stable on the space  $X$**  if it is stable for every problem  $\theta$ :  $\theta \in \Theta_X$ .

**Definition 3.** We shall say that the space  $X$  **satisfies condition (A1)** if from the fact that  $\{x_n\} \subset X$ ,  $x_0 \in X$ ,

$$x_n \xrightarrow{w} x_0 \quad \text{and} \quad \|x_n\| = \inf_{x \in \overline{\text{co}}(x_n, x_{n+1}, \dots)} \|x\|$$

it follows that  $x_n \rightarrow x_0$ .

**Definition 4.** We shall say that the space  $X$  **satisfies condition (A2)** if from the fact that  $\{x_n\} \subset X$ ,  $x_0 \in X$ ,  $x_n \xrightarrow{w} x_0$ ,  $\|x_n\| = 1$  and

$$\lim_{n \rightarrow \infty} \sup_{k \geq n} \rho(x_k, H_n) = 0,$$

where  $H_n = \{x : f_n(x) = 1, f_n \in X^*, \|f_n\| = 1\}$  is a closed hyperplane supporting the ball  $s = \{x : \|x\| \leq 1\}$  at the point  $x_n$ , it follows that  $x_n \rightarrow x_0$ .

**Definition 5.** We shall say that the space  $X$  **satisfies condition (A3)** if from the fact that  $\{x_n\} \subset X$ ,  $x_0 \in X$ ,  $\|x_n\| = 1$ ,

$$x_n \xrightarrow{w} x_0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \rho(x_0, H_n) = 0,$$

where  $H_n$  is a closed hyperplane supporting the ball  $s = \{x : \|x\| \leq 1\}$  at the point  $x_n$ , it follows that  $x_n \rightarrow x_0$ .

**Definition 6.** We shall say that the space  $X$  satisfies condition (A4) if from the fact that  $\{x_n\} \subset X$ ,  $x_0 \in X$ ,  $\|x_n\| = \|x_0\| = 1$ , and  $x_n \xrightarrow{w} x_0$ , it follows that  $x_n \rightarrow x_0$ .

**Definition 7.** The space  $X$  is called an  $E$ -space (see (9)) if it is reflexively strictly convex and satisfies condition (A4).  $E$ -spaces are the largest known class of Banach spaces for which the operator  $P$  of metric projection onto a closed convex set is well-posed in the sense of Hadamard (see (6)).

#### 4. Main theorems.

**Theorem 1.** Let  $X$  be a separable reflexive strictly convex space. In order that the residual method be stable on the space  $X$ , it is necessary that  $X$  satisfy condition (A1).

**Theorem 2.** Let  $X$  be a reflexive strictly convex space. In order that the residual method be stable on the space  $X$ , it is sufficient that  $X$  satisfy condition (A4).

**Theorem 3.** Let  $X$  be a reflexive strictly convex space; then conditions (A1), (A2), (A3), (A4) on the space  $X$  are equivalent.

**Theorem 4.** Let  $X$  be a separable reflexive strictly convex space. In order that the residual method be stable on the space  $X$ , it is necessary and sufficient that  $X$  be an  $E$ -space.

**5. Finite-dimensional approximations of problem  $\theta$ .** Let  $X$  be a separable  $E$ -space and

$$X_1 \subseteq X_2 \subseteq \dots \subseteq X_n \subseteq \dots \subseteq X \quad (5)$$

be an increasing chain of finite-dimensional subspaces of the space  $X$  such that

$$\overline{\bigcup_{n=1}^{\infty} X_n} = X. \quad (6)$$

The existence of such an increasing chain follows easily from the separability of  $X$ . We shall solve the problem of finding a point of minimum of the functional

$$K[x] = \|x\| \quad (7)$$

under the conditions

$$x \in X_n, \quad Ax \in V_\delta. \quad (8)$$

From the fact that  $X$  is an  $E$ -space, there follows the existence and uniqueness of the point of minimum. Denote this point by  $x_\delta^n$ .

**Theorem 5.** The sequence of elements  $\{x_\delta^n\}$  converges strongly to the approximate solution  $x_\delta$  of problem  $\theta$ :

$$x_{\delta}^n \rightarrow x_{\delta} \quad \text{as } n \rightarrow \infty. \quad (9)$$

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Received  
25 VIII 1969

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