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Abstract

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MATHEMATICS

S. Z. SHEFEL'

ISOMETRIC IMMERSIONS OF CLASS C^1

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An immersion in Euclidean space $\mathbf{R}^3$ of a two-dimensional Riemannian manifold by a surface F of class C^1 or even $C^{1,\alpha}$ for small α , as follows from works (¹⁻³), may be such that for the surface F the relations between its intrinsic and extrinsic geometry that are characteristic of surfaces of class C^2 are violated. Thus, for example, a closed surface with positive intrinsic curvature need not be convex, a surface with negative curvature need not be saddle-shaped, and a surface isometric to the plane need not be a cylinder.

In this connection it makes sense to consider isometric immersions in $\mathbf{R}^3$ of class C^1 subject to certain additional restrictions.

Such restrictions include the condition that the extrinsic curvature of the surface be bounded (⁴) and the condition that the surface belong to the class $C^{1,\alpha}$ for $\alpha > 2/3$ (⁵).

In the present note, the condition of $\mathfrak{A}$ -regularity is imposed on isometric immersions by a surface of class C^1 .

Definition. An isometric immersion of a metric of class K by a surface F in $\mathbf{R}^3$ is called **regular** in the class K with respect to a group of affine transformations $\mathfrak{A}$ of the space $\mathbf{R}^3$ ($\mathfrak{A}$ -regular) if, under any affine transformation of $\mathbf{R}^3$, the surface F is transformed into a surface that is an isometric immersion of some metric of the same class K .

Immersions in $\mathbf{R}^n$ regular with respect to the group of transformations of $\mathbf{R}^n$ were considered in (⁶).

We denote by A, A^+, A^-, A^0 , respectively, the classes of metrics of bounded curvature, nonnegative, nonpositive curvature, and zero curvature (the class of locally Euclidean metrics).

Theorems 1-3 establish structural properties of $\mathfrak{A}$ -regular immersions of metrics of the classes A^+, A^-, A^0 . Theorems 4-6 establish connections between the conditions of bounded extrinsic curvature, $\mathfrak{A}$ -regularity, and membership of the surface in the class $C^{1,\alpha}$ for $\alpha > 2/3$. At the same time one should additionally note that the immersion of a metric of any of the classes A, A^+, A^-, A^0 by a

surface of bounded extrinsic curvature is, obviously, $\mathfrak{A}$ -regular. Theorem 6 is a generalization of an analogous theorem of Yu. F. Borisov ⁽⁵⁾.

Theorem 1. *If a surface $F \in C^1$ is an $\mathfrak{A}$ -regular immersion in $\mathbb{R}^3$ of a metric of class A^+ , then for every point M of it at least one of the following assertions is true:*

(α). *Some neighborhood of the point M on the surface F is a convex surface (in particular, this neighborhood may be a plane domain).*

(β). *Through the point M there passes a rectilinear segment $\pi(M)$ lying on the surface F , its ends are situated on the boundary of F , and the tangent plane to F along the segment $\pi(M)$ is stationary. If the point M has no plane neighborhood on the surface F , then such a segment is unique and none of its points has a plane neighborhood on F .*

Theorem 1'. If a surface F satisfies the conditions of Theorem 1, then every cap cut from the surface F by a plane is a convex surface.

Theorem 2. If a surface F is an $\mathfrak{A}$ -regular immersion in $\mathbb{R}^3$ of a metric of class A^- , then F is a saddle surface.

Theorem 3. If a surface F in $\mathbb{R}^3$ of class C^1 is an $\mathfrak{A}$ -regular immersion of a metric of class A^0 , then for any of its points M at least one of the following assertions is true:

(α). Some neighborhood of the point M on the surface F is a planar domain.

(β). Through the point M there passes a rectilinear segment $\pi(M)$, lying on the surface F , with endpoints on the boundary of F ; the tangent plane to F along $\pi(M)$ is stationary. If the point M has no planar neighborhood on the surface F , then such a segment is unique and none of its points has a planar neighborhood on F .

Theorem 4. If a surface F of class $C^{1,\alpha}$ ($\alpha > 1/2$) is an $\mathfrak{A}$ -regular immersion in $\mathbb{R}^3$ of a metric of one of the classes A^+ , A^- , A^0 , then F is a surface of bounded extrinsic curvature.

Theorem 5. If a surface F of class $C^{1,\alpha}$ ($\alpha > 2/3$) is an immersion in $\mathbb{R}^3$ of a metric of one of the classes A , A^+ , A^- , A^0 , then this immersion F is an $\mathfrak{A}$ -regular immersion.

From Theorems 1-5 it follows that

Theorem 6. If a surface $F \in C^{1,\alpha}$ ($\alpha > 2/3$) is an immersion in $\mathbb{R}^3$ of a metric of one of the classes A^+ , A^- , A^0 , then it is a surface of bounded extrinsic curvature and for it the assertion of the corresponding structural Theorem 1, 2, or 3 is satisfied.

The **proof** of Theorem 1 is based on the following lemmas.

Lemma 1. If a surface F satisfies the conditions of Theorem 1, is given by the equation $z = f(x, y)$, $(x, y) \in K$, where K is a closed domain in the x, y -plane,

and if at the interior points of K $f(x, y) > 0$, while on the boundary of K $f(x, y) = 0$, then F is a convex surface.

The central point in the proof of Lemma 1 is the following assertion: some neighborhood of the "vertex" of any cap cut by a plane from the surface F is a convex surface.

Lemma 2. If a surface $F \in C^1$ is saddle and is an isometric immersion of a metric of class A^0 , then for it the assertions of Theorem 2 are fulfilled.

Proof. Introduce on the surface F a new topology induced by the Gauss map of F onto the sphere. Divide F into two sets F^0 and $F' = F \setminus F^0$, where F^0 is the set of points of F interior with respect to the topology introduced above. If F^0 is nonempty, then the extrinsic curvature of F^0 coincides with the intrinsic curvature, which is proved by approximating the surface F^0 by a C^1 -sequence of saddle polyhedra. Hence, by a theorem of A. V. Pogorelov ⁽⁴⁾, it follows that for the points of F^0 the assertions of Theorem 2 are fulfilled.

The proof of Theorem 3 is given in ⁽⁶⁾.

The proof of Theorem 4 is carried out analogously to the proof of Lemma 2. We note only that from the condition $F \in C^{1,\alpha}$ ($\alpha > 1/2$) it follows that the image of the set F' on the sphere has measure zero.

The proof of Theorem 5 is based on the work ⁽⁷⁾. Let $F \in C^{1,\alpha}$ ($\alpha > 2/3$), and let the intrinsic metric of F belong to class A . Define on F a set function σ , completely additive on the ring of Borel sets, which on each domain $G \subset F$ with rectifiable boundary L coincides with the quantity $\Delta(L')$, where L' is the spherical image of L , and Δ is the oriented area of L' .

Let α be an affine transformation of $\mathbb{R}^3$, and let G be a domain on F bounded by a smooth curve.

If the variation $|\sigma(F)| < \infty$, then the formula holds

$$\sigma(\alpha(G)) = \int_G J(x') \sigma(dG),$$

where x' is the spherical image of the point $x \in G$, and $J(x')$ is the Jacobian of the diffeomorphism β of the unit sphere S onto itself, defined by the equality $x' = \beta(\alpha(x'))$, for all $x' \in S$. It follows from ⁽⁷⁾ that in order for the surface $F \in C^1$, ($\alpha > 1/2$), to be an isometric immersion of a metric of class A , it is necessary and sufficient that the variation $|\sigma(F)| < \infty$. In this case the intrinsic curvature $\omega(F) \equiv \sigma(F)$.

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Institute of Mathematics Siberian Branch of the Academy of Sciences of the USSR Novosibirsk

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