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Abstract

Full Text

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PHYSICS

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ON SPATIALLY HOMOGENEOUS GRAVITATIONAL FIELDS

(Presented by Academician Ya. B. Zel'dovich on 28 V 1969)

We shall call a metric spatially homogeneous (s.h.) if it admits a simply or multiply transitive group of motions $\tilde{G}_r$ ($r \geq 3$), acting on spacelike hypersurfaces. Such a definition is often encountered in the literature ⁽¹⁻³⁾, although it is not entirely justified from the physical point of view ⁽³⁾. Let us consider the case $r = 3$. Metrics satisfying the stated requirements are known. Usually one chooses a semigeodesic (synchronous) coordinate system in which the hypersurfaces of transitivity have equations $x^0 = \text{const}$, and each component of the Killing vectors ξ^i is at most a function of x^1, x^2, x^3 . Then the metrics obtained as a result of integrating the Killing equations can be written in the form ^{(1,2)*}

$$ds^2 = dx^{02} + g_{ik} dx^i dx^k = c^2 dt^2 + \alpha_{ab} e_i^a e_k^b dx^i dx^k.$$

Here α_{ab} are arbitrary functions of t , and e_i^a are Killing vectors of the group $\tilde{G}_3$, reciprocal to the given ones ⁽⁴⁾. The quantities e_i^a are defined by the relations $e_i^a e_a^k = \delta_i^k$, $e_i^a e_b^i = \delta_b^a$. For each of the 9 types of real nonisomorphic structures ^(5,7), e_i^a can be chosen so that the structure constants $\bar{C}_{bc}^a$ from the relations

$$e_{i,k}^a - e_{k,i}^a = \bar{C}_{bc}^a e_i^b e_k^c \quad (1)$$

coincide with the structure constants C_{bc}^a of the given G_3 ($\xi_{b,k}^i \xi_c^k - \xi_{c,k}^i \xi_b^k = C_{bc}^a \xi_a^i$). The form of the operators ξ_a^i and e_i^a is given in the Appendix.

S.h. metrics satisfying the Einstein equations $R_{\mu\nu} = -\kappa(T_{\mu\nu} - 1/2 g_{\mu\nu} T)$ are being studied intensively in connection with their cosmological applications. As the energy-momentum tensor of the medium one most often takes

$$T_{\mu\nu} = (\rho c^2 + p) u_\mu u_\nu - p g_{\mu\nu}, \quad (2)$$

where ρc^2 and p are the energy density and pressure; u_μ are the components of the velocity of the medium, which in the present case have the form ⁽²⁾

$u_0 = n_0(t)$, $u_i = n_a(t)e_i^a$. Generally speaking, u^i are not equal to zero, i.e. the synchronous coordinate system used is not comoving with the medium. However, in cosmological problems we are primarily interested in the behavior of the medium—its density, deformation, rotation, etc.; for this it is convenient to use a comoving reference system (c.r.s.). And in those cases where one seeks solutions satisfying previously imposed requirements (for example, the requirement of a specified dependence of the density on the proper time τ of an element of the medium—

* Greek indices $\mu, \nu, \dots$ take the values 0, 1, 2, 3; Latin $i, j, k, \dots$, 1, 2, 3. “Frame” indices $a, b, c, d, \dots$ take the values 1, 2, 3. A comma with an index denotes partial differentiation.

which is important in calculating the chemical composition of matter), it is practically impossible to do without an s.s.r. The aim of the article is to derive the Einstein equations in an s.s.r. for all types of spatially homogeneous metrics.

There exist coordinate transformations (containing 4 arbitrary functions of t) that leave invariant the form of ξ_i^a and the equation $x^0 = \text{const}$. Having found these transformations, it is easy to show that, by a choice of the arbitrary functions, all u^i can be made equal to zero, i.e., an s.s.r. can be introduced. The synchronicity of the coordinate system is thereby violated. Suppose that the choice of s.s.r. has been made, and integrate Killing’s equations. As a result we obtain

$$g_{00} = \alpha_{00}, \quad g_{0i} = \alpha_{0a}e_i^a, \quad g_{ik} = \alpha_{ab}e_i^a e_k^b, \quad (3)$$

where the α' ’s are arbitrary functions of t . In view of the possibility of direct physical interpretation of the quantities entering the Einstein equations, we write these equations in the chronometrically invariant (ch.i.) form proposed by Zelmanov (⁶):

$$\begin{aligned} {}^* \dot{D} + D_{ik}D^{ik} + A_{ik}A^{ki} + {}^* \nabla_i F^i - F_i F^i &= -\frac{1}{2}(\rho + U), \\ {}^* \dot{D}_{ik} - (D_{ij} + A_{ij})(D_k^j + A_k^j) + DD_{ik} - D_{ij}D_k^j + 3A_{ij}A_k^j + \\ + \frac{1}{2}({}^* \nabla_i F_k + {}^* \nabla_k F_i) - F_i F_k - C_{ik} &= \frac{1}{2}(\rho h_{ik} + 2U_{ik} - U h_{ik}), \\ {}^* \nabla_k (h^{ik}D - D^{ik} - A^{ik}) + 2F_k A^{ik} &= J^i, \end{aligned} \quad (4)$$

The constants c and $\varkappa$ are equal to 1; a dot denotes $\frac{\partial}{\partial t}$, while a dot with an asterisk denotes $^*\frac{\partial}{\partial t}$. Ch.i. quantities and operators (marked by $*$) are expressed in terms of the ordinary operators and the metric $g_{\mu\nu}$ as follows:

$$^*\frac{\partial}{\partial t} = \frac{1}{\sqrt{g_{00}}} \frac{\partial}{\partial t}; \quad ^*\frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^i} - \frac{g_{0i}}{g_{00}} \frac{\partial}{\partial t};$$

the metric tensor

$$h_{ik} = -g_{ik} + \frac{g_{0i}g_{0k}}{g_{00}};$$

the tensor of deformation velocities

$$D_{ik} = \frac{1}{2} {}^*\dot{h}_{ik};$$

the vector of gravitational-inertial force

$$F_i = (1-w)^{-1}(w_{,i} - \dot{V}_i);$$

the tensor of angular velocity of rotation

$$A_{ik} = \frac{1}{2}(V_{k,i} - V_{i,k}) + \frac{1}{2}(F_i V_k - F_k V_i),$$

where the auxiliary quantities w, V_i are defined by $g_{00} = (1-w)^2$, $g_{0i} = -V_i(1-w)$. Further, the Ricci tensor of space is

$$C_{lk} = H_{lk} - A_{ki} D_l^i - A_{li} D_k^i - A_{kl} D, \quad \text{where } H_{lk} = H_{lki}{}^i;$$

$$H_{lki}{}^j Q_j = (^*\nabla_{ik} - ^*\nabla_{ki})Q_l - 2A_{ik}{}^* \dot{Q}_l; \quad (5)$$

Q_l is an arbitrary vector; $^*\nabla_i$ is the operator of covariant differentiation, constructed according to the usual rules from the metric h_{ik} and $^*\frac{\partial}{\partial x^i}$. Finally,

$$\rho = T_{00}(g_{00})^{-1}, \quad J^i = T_0^i(g_{00})^{-1/2}, \quad U^{ik} = T^{ik}.$$

For the tensor (2), and in an s.s.r.,

$$J^i = 0, \quad U^{ik} = p h^{ik}.$$

For spatially homogeneous metrics (3), all the vector and tensor quantities considered can be “expanded” with respect to the frame vectors e_i^a , with coefficients depending only on t . For example,

$$h_{ik} = \left(-\alpha_{ab} + \frac{\alpha_{0a}\alpha_{0b}}{\alpha_{00}} \right) e_i^a e_k^b = \gamma_{ab}(t) e_i^a e_k^b.$$

With the help of γ_{ab} , frame indices are shifted. We denote the frame components of the quantities $D_{ik}, V_i, F_i, A_{ik}, C_{lk}$ by the corresponding lowercase letters; then

$$d_{ab} = \frac{1}{2} {}^* \dot{\gamma}_{ab}, \quad v_a = -\frac{\alpha_{0a}}{\sqrt{\alpha_{00}}}, \quad f_a = -{}^* \dot{v}_a,$$

and, taking (1) into account,

$$a_{ab} = \frac{1}{2} v_c C^c{}_{ba} + \frac{1}{2} (f_a v_b - f_b v_a).$$

We find how c_{ab} is expressed in terms of $\gamma_{ab}, v_a, C^a{}_{bc}$. In (5), as Q_i we substitute e_i^d . We introduce the notation $-\Gamma_{ac}^d =$

$$= e_a^i e_c^{k*} \nabla_k e_i^d.$$

Then

$$e_a^i e_b^j e_c^{k*} \nabla_j e_{ki}^d = -\Gamma_{ac\oplus b}^d + \Gamma_{gc}^d \Gamma_{ab}^g + \Gamma_{ag}^d \Gamma_{cb}^g,$$

where

$$\Gamma_{ac\oplus b}^d = e_b^{j*} \nabla_j \Gamma_{ac}^d.$$

Since

$${}^*(e_l^d)^\bullet = 0,$$

from (5) we obtain

$$H_{ab} = H_{abc}{}^c = -2\Gamma_{a[c\oplus b]}^c + 2\Gamma_{d[b}^c \Gamma_{a]c}^d + 2\Gamma_{ad}^c \Gamma_{[bc]}^d, \quad (6)$$

where $X_{[a|b|c]} = \frac{1}{2}(X_{abc} - X_{cba})$. The required c_{ab} are related to H_{ab} by

$$c_{ab} = H_{ab} - a_{bc} d_a{}^c - a_{ac} d_b{}^c - a_{ba} d. \quad (7)$$

Let us compute the quantities Γ_{ab}^c . Applying the operator ${}^* \nabla_k$ to $\gamma_{ab} = e_a^i e_{bi}$, we get

$$\Gamma_{abc} + \Gamma_{bac} = v_c^* \gamma_{ab},$$

and from the definition of Γ_{ac}^d and (1) it follows that

$$\Gamma_{bc}^d - \Gamma_{cb}^d = C^d{}_{cb}.$$

If one uses the equalities obtained from these by a cyclic interchange of indices, one can find

$$-\Gamma_{abc} = \frac{1}{2}(C_{abc} + C_{bca} - C_{cab} - v_c^* \gamma_{ab} - v_b^* \gamma_{ac} + v_a^* \gamma_{bc}).$$

After minor transformations,

$$-\Gamma_{ab}^c = \frac{1}{2} [C_{ab}^c + \gamma^{cd}(\gamma_{ag} C_{bd}^g + \gamma_{bg} C_{ad}^g)] - v_a d_b^c - v_b d_a^c + v^c d_{ab}. \quad (8)$$

Since Γ_{ab}^c depend only on t , we have

$$\Gamma_{ab \oplus d}^c = v_d^* (\Gamma_{ab}^c)^\bullet,$$

and, taking this fact into account, formulas (6)–(8) solve the problem posed. In the Appendix are given $\bar{c}_{ab}$, computed by (6)–(8) under the condition $a_{00} = 1$, $a_{0a} = 0$. Let us also introduce the notation

$${}^* \nabla_i = e_i^c \square_c.$$

It is not difficult to prove that for any quantity

$$X_{i\dots}^{\dots k} = x_{a\dots}^{\dots b}(t) e_i^a \dots e_b^k$$

one has

$${}^* \nabla_j X_{i\dots}^{\dots k} = e_j^c e_i^a \dots e_b^k \square_c x_{a\dots}^{\dots b},$$

where

$$\square_c x_{a\dots}^{\dots b} = v_c^* (x_{a\dots}^{\dots b})^\bullet - x_{d\dots}^{\dots b} \Gamma_{ac}^d - \dots + x_{a\dots}^{\dots d} \Gamma_{dc}^b.$$

Finally, let us pass to proper time by means of

$$d\tau = \sqrt{a_{00}} dt$$

(which is equivalent to the choice $a_{00} = 1$) and write equations (4) in the s.s.o. in the form of a system of ordinary differential equations for functions of τ ($d/d\tau$ is denoted by a dot):

$$\begin{aligned} \dot{d} + d_{ab} d^{ba} + a_{ab} a^{ba} + \square_a f^a - f_a f^a &= -\frac{1}{2}(\rho + 3p), \\ \square_b (\gamma^{ab} d - d^{ab} - a^{ab}) + 2f_b a^{ab} &= 0, \\ \dot{d}_{ab} - (d_{ac} + a_{ac})(d_b^c + a_b^c) + d d_{ab} - d_{ac} d_b^c + 3a_{ac} a_b^c \\ + \frac{1}{2}(\square_a f_b + \square_b f_a) - f_a f_b - c_{ab} &= \frac{1}{2}(\rho - p) \gamma_{ab}. \end{aligned} \quad (9)$$

Let us note that from the hydrodynamic equations

$$\dot{\rho} + d(\rho + p) = 0, \quad v_a \dot{p} = (\rho + p) f_a,$$

which are a consequence of (9), it follows that

$$f_a v_b - f_b v_a = 0,$$

i.e.

$$a_{ab} = \frac{1}{2} v_c C^c_{ba}.$$

Moreover,

$$v_a = -a_{0a} = k_a \exp \int \frac{dp}{\rho + p},$$

where k_a are arbitrary constants.

The author thanks A. L. Zelmanov for discussion.

Appendix

Introduce the notation

$$X_a = \xi_a^i \frac{\partial}{\partial x^i} = \xi_a^i p_i;$$

denote the determinant of the matrix γ_{ab} by γ ; then:

Type I.

$$X_1 = p_1, \quad X_2 = p_2, \quad X_3 = p_3, \quad e_i^1 = (1, 0, 0), \quad e_i^2 = (0, 1, 0), \quad e_i^3 = (0, 0, 1), \quad \bar{c}_{ab} = 0.$$

Type II.

$$X_1 = p_1, \quad X_2 = p_2, \quad X_3 = x^2 p_1 - p_3, \quad e_i^1 = (1, x^3, 0), \quad e_i^2 = (0, 1, 0),$$

$$e_i^3 = (0, 0, 1), \quad \bar{c}_1^2 = \bar{c}_1^3 = \bar{c}_2^3 = 0, \quad -\bar{c}_1^1 = \bar{c}_2^2 = \bar{c}_3^3 = \frac{1}{2\gamma} \gamma_{11}^2.$$

Type III.

$$X_1 = p_1, \quad X_2 = p_2, \quad X_3 = x^1 p_1 - p_3, \quad e_i^1 = (e^{x^3}, 0, 0), \quad e_i^2 = (0, 1, 0),$$

$$e_i^3 = (0, 0, 1), \quad \bar{c}_1^3 = \bar{c}_2^3 = 0, \quad \bar{c}_1^2 = -\frac{1}{\gamma} \gamma_{11} \gamma_{12}, \quad \bar{c}_1^1 = \bar{c}_3^3 = \gamma^{33} + \frac{1}{2\gamma} \gamma_{12}^2, \quad \bar{c}_2^2 = -\frac{1}{2\gamma} \gamma_{12}^2.$$

$$\begin{aligned} \text{Type IV. } X_1 &= p_1, \quad X_2 = p_2, \quad X_3 = (x^1 + x^2) p_1 + x^2 p_2 - p_3, \quad e_i^1 = \\ &= (e^{x^3}, x^3 e^{x^3}, 0), \quad e_i^2 = (0, e^{x^3}, 0), \quad e_i^3 = (0, 0, 1), \quad \bar{c}_1^3 = \bar{c}_2^3 = 0, \quad \bar{c}_1^2 = \frac{1}{\gamma} \gamma_{11}^2, \quad \bar{c}_1^1 = \\ &= 2\gamma^{33} - \frac{1}{2\gamma} \times \gamma_{11}(\gamma_{11} + 2\gamma_{12}), \quad \bar{c}_2^2 = 2\gamma^{33} + \frac{1}{2\gamma} \gamma_{11}(\gamma_{11} + 2\gamma_{12}), \quad \bar{c}_3^3 = 2\gamma^{33} + \frac{1}{2\gamma} \gamma_{11}^2. \end{aligned}$$

$$\text{Type V. } X_1 = p_1, \quad X_2 = p_2, \quad X_3 = x^1 p_1 + x^2 p_2 - p_3, \quad e_i^1 = (e^{x^3}, 0, 0), \quad e_i^2 = (0, e^{x^3}, 0), \quad e_i^3 = (0, 0, 1), \quad \bar{c}_{ab} = 2\gamma^{33} \gamma_{ab}.$$

Type VI. $X_1 = p_1$, $X_2 = p_2$, $X_3 = x^1 p_1 + qx^2 p_2 - p_3$, $e_i^1 = (e^{x^3}, 0, 0)$, $e_i^2 = (0, e^{qx^3}, 0)$, $e_i^3 = (0, 0, 1)$, $\bar{c}_1^3 = \bar{c}_2^3 = 0$, $\bar{c}_1^2 = \frac{1}{\gamma} \gamma_{11} \gamma_{12} (q - 1)$, $\bar{c}_1^1 = \frac{1}{2\gamma} \gamma_{12}^2 \times (1 - q^2) + (1 + q) \gamma^{33}$, $\bar{c}_2^2 = -\frac{1}{2\gamma} \gamma_{12}^2 (1 - q)^2 + q(1 + q) \gamma^{33}$, $\bar{c}_3^3 = \frac{1}{2\gamma} \gamma_{12}^2 (1 - q)^2 + (1 + q^2) \gamma^{33}$, where $q \neq 0, 1$.

Type VII. $X_1 = p_1$, $X_2 = p_2$, $X_3 = -x^2 p_1 + (qx^2 + x^1) p_2 + p_3$, $e_i^1 = [e^{-\frac{1}{2}qx^3} (q \sin \frac{p}{2} x^3 + p \cos \frac{p}{2} x^3), 2e^{-\frac{1}{2}qx^3} \sin \frac{p}{2} x^3, 0]$, $e_i^2 = [-2e^{-\frac{1}{2}qx^3} \times \sin \frac{p}{2} x^3, e^{-\frac{1}{2}qx^3} (-q \sin \frac{p}{2} x^3 + p \cos \frac{p}{2} x^3), 0]$, $e_i^3 = (0, 0, 1)$, $q^2 < 4$, $p = \sqrt{4 - q^2}$, $\bar{c}_1^1 = \frac{1}{2\gamma} [\gamma_{22}^2 - (\gamma_{11} - q\gamma_{12})^2]$, $\bar{c}_2^2 = -\frac{1}{2\gamma} [\gamma_{22}^2 + (\gamma_{11} + q\gamma_{12})^2 - 2\gamma_{11}(\gamma_{11} + q\gamma_{22})]$, $\bar{c}_3^3 = \frac{1}{2\gamma} [-(\gamma_{11} + \gamma_{22} + q\gamma_{12})^2 + 2(\gamma_{11}^2 + \gamma_{22}^2 + 2\gamma_{12}^2) + 2q^2 \gamma_{11} \gamma_{22}]$, $\bar{c}_1^3 = \bar{c}_2^3 = 0$, $\bar{c}_1^2 = -\frac{1}{\gamma} [\gamma_{12}(\gamma_{11} + \gamma_{22}) - q\gamma_{11} \gamma_{22}]$.

Type VIII. $X_1 = p_2$, $X_2 = x^2 p_2 + p_3$, $X_3 = e^{x^3} p_1 + x^{22} p_2 + 2x^2 p_3$, $e_i^1 = (1, x^1 e^{-x^3}, -x^1)$, $e_i^2 = (0, -2x^1 e^{-x^3}, 1)$, $e_i^3 = (0, e^{-x^3}, 0)$, $\bar{c}_1^2 = -2\bar{c}_2^3 = -4 \left(\gamma^{23} + \frac{1}{\gamma} \gamma_{12} A \right)$, $\bar{c}_1^3 = -2 \left(2\gamma^{33} - \frac{1}{\gamma} \gamma_{11} A \right)$, $\bar{c}_1^1 = \bar{c}_3^3 = -2 \left(2\gamma^{13} - \frac{1}{\gamma} \gamma_{22} A \right)$, $\bar{c}_2^2 = 2 \left[\gamma^{22} - \frac{1}{\gamma} (\gamma_{22} + \gamma_{13}) A \right]$, where $A = \gamma_{22} - \gamma_{13}$.

Type IX. $X_1 = p_2$, $X_2 = \cos x^2 p_1 - \text{ctg } x^1 \sin x^2 p_2 + \frac{\sin x^2}{\sin x^1} p_3$, $X_3 = -\sin x^2 p_1 - \text{ctg } x^1 \cos x^2 p_2 + \frac{\cos x^2}{\sin x^1} p_3$, $e_i^1 = (\cos x^3, \sin x^3 \sin x^1, 0)$, $e_i^2 = (-\sin x^3, \cos x^3 \sin x^1, 0)$, $e_i^3 = (0, \cos x^1, 1)$, $\bar{c}_1^2 = -2\gamma^{12} - \frac{1}{\gamma} \gamma_{12} A$, $\bar{c}_1^1 = \frac{1}{2\gamma} A^2 - B - \gamma^{11} - \frac{1}{\gamma} (\gamma_{11}^2 + \gamma_{12}^2 + \gamma_{13}^2)$, where $A = \gamma_{11} + \gamma_{22} + \gamma_{33}$, $B = \gamma^{11} + \gamma^{22} + \gamma^{33}$; the remaining components $\bar{c}_a^b$ are obtained by a cyclic replacement of the indices 1, 2, 3.

For some geometrical properties of the metrics considered, see ^(3, 8).

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