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Abstract

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MATHEMATICS

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ON ONE VARIANT OF AVERAGING IN INTEGRO-DIFFERENTIAL EQUATIONS

(Presented by Academician A. A. Dorodnitsyn on 17 XI 1969)

1. Consider the system of nonlinear integro-differential equations

$$\dot{x} = \varepsilon X \left(t, x, \int_0^t \varphi(t, s, x(s)) ds \right). \quad (1)$$

Here $\varepsilon > 0$ is a small parameter, $x = (x_1, x_2, \dots, x_n)$.

In papers ^(1, 2), for equations of the form (1) the following averaging procedure was proposed. Let

$$\psi(t, x) = \int_0^\infty \varphi(t, s, x) ds, \quad (2)$$

$$X_0(x) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T X(t, x, \psi(t, x)) dt. \quad (3)$$

Then to the system (1) there is put in correspondence an averaged system of the form

$$\dot{\xi} = \varepsilon X_0(\xi). \quad (4)$$

In the present note theorems are proved on the closeness of the solutions of systems (1) and (4), both on a finite and on an infinite interval.

Theorem 1. Let the functions $X(t, x, y)$ and $\varphi(t, s, x)$ be defined and continuous in the domain $Q\{t \geq 0, s \geq 0, x \in D, y \in E_n\}$, and suppose that in this domain the following conditions are fulfilled:

1)

$$|X(t, x', y') - X(t, x'', y'')| \leq \lambda\{|x' - x''| + |y' + y''|\},$$

$$|\varphi(t, s, x') - \varphi(t, s, x'')| \leq \mu(t, s)|x' - x''|,$$

$$\int_0^t d\tau \int_0^\tau \mu(\tau, s) ds \leq At^\alpha, \quad A > 0, \quad 0 \leq \alpha < 1, \quad \lambda = \text{const.}$$

2) The function $\psi(t, x)$, defined by equality (2), satisfies the Lipschitz condition

$$|\psi(t, x') - \psi(t, x'')| \leq \nu|x' - x''|, \quad \nu = \text{const.}$$

3) At each point x of the domain D the limit (3) exists, and the function $X_0(x)$ is bounded ($|X_0| \leq M$) and satisfies the Lipschitz condition.

4) The solution $\xi = \xi(t)$, $\xi(0) = x(0) \in D$, of the averaged system is defined for all $t \geq 0$ and lies in the domain D together with some ρ -neighborhood.

5) Along the trajectory $\xi(t)$,

$$\int_0^t d\tau \left| \int_\tau^\infty \varphi(\tau, s, \xi(\tau)) ds \right| \leq Bt^\beta, \quad B > 0, \quad 0 \leq \beta < 1.$$

Then for any $\eta > 0$ and $L > 0$ one can specify an ε_0 such that, for $\varepsilon < \varepsilon_0$, on the interval $0 \leq t \leq L\varepsilon^{-1}$ the inequality

$$|x(t) - \xi(t)| < \eta$$

will hold.

Proof. As in (3), it is shown that for any $a > 0$ one can specify an $\bar{\varepsilon}$ such that, for $\varepsilon < \bar{\varepsilon}$, on the interval $0 \leq t \leq L\varepsilon^{-1}$ the inequality

$$\varepsilon \left| \int_0^t \left[X\left(\tau, \xi(\tau), \int_0^\infty \varphi(\tau, s, \xi(\tau)) ds\right) - X_0(\xi(\tau)) \right] d\tau \right| < a$$

will hold. Hence,

$$\begin{aligned} |x - \xi| &\leq a + \varepsilon \lambda \int_0^t |x(\tau) - \xi(\tau)| d\tau + \varepsilon \lambda \int_0^t d\tau \int_0^\tau \mu(\tau, s) |x(s) - \xi(s)| ds \\ &\quad + \varepsilon \lambda \int_0^t d\tau \left| \int_\tau^\infty \varphi(\tau, s, \xi(\tau)) ds \right| + \varepsilon \lambda \int_0^t d\tau \int_0^\tau \mu(\tau, s) |\xi(s) - \xi(\tau)| ds. \end{aligned}$$

From this, taking into account the conditions of the theorem, we find on the interval $0 \leq t \leq L\varepsilon^{-1}$:

$$|x - \xi| \leq (a + \lambda M A L^{1+\alpha} \varepsilon^{1-\alpha} + \lambda B L^{1+\beta} \varepsilon^{1-\beta}) e^{\lambda L + A L \alpha \varepsilon^{1-\alpha}}.$$

The assertion of the theorem follows from the last inequality.

Remark 1. Consider the system

$$\dot{x} = \varepsilon X(t, x) + \varepsilon \int_0^t Y(t, s, x(s)) ds. \quad (*)$$

If the functions X and Y are bounded, then the solution $x(t)$ of system $(*)$ will change substantially already on an interval of order $\varepsilon^{-1/2}$. Therefore, in particular, for systems of the form $(*)$ one can formulate an averaging theorem analogous to Theorem 1, establishing the closeness of the solutions of the original and averaged systems on the interval $0 \leq t \leq L\varepsilon^{-1/2}$. In this case the parameters α and β will vary within the limits $0 \leq \alpha < 2$, $0 \leq \beta < 2$.

Theorem 2. Replace condition (3) of Theorem 1 by the following:

- a) at every point x of the domain D , uniformly with respect to t , there exists the limit

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_t^{t+T} X(t, x, \psi(t, x)) dt = X_0(x), \quad (5)$$

and the function $X_0(x)$ is bounded and satisfies the Lipschitz condition;

- b) the solution $\xi = \xi(\tau)$, $\tau = \varepsilon t$, of the averaged system

$$\dot{\xi} = \varepsilon X_0(\xi), \quad \xi(0) = x(0)$$

is uniformly* asymptotically stable;

- c) equation (1) has no singular points.

Then for any $0 < \eta < \rho$ one can specify an ε_0 such that, for $\varepsilon < \varepsilon_0$, for all $t > 0$ the inequality

$$|x(t) - \xi(t)| < \eta$$

will hold.

Proof. The proof is carried out by the methods set forth in (4, 6).

* For the notion of uniform asymptotic stability, see, for example, (5).

Remark 2. Let us note that in many cases the requirement of uniform asymptotic stability appearing in Theorem 2 can be weakened.

2. Consider a system of a more general form

$$\dot{x} = \varepsilon X \left(t, x, \dot{x}, y, \dot{y}, \int_0^t \varphi(t, s, x(s), \dot{x}(s), y(s), \dot{y}(s)) ds \right),$$

$$\dot{y} = Y_0(t, x, y) + \varepsilon Y_1 \left(t, x, \dot{x}, y, \dot{y}, \int_0^t \psi(t, s, x(s), \dot{x}(s), y(s), \dot{y}(s)) ds \right). \quad (6)$$

If the general solution of system (6) for $\varepsilon = 0$ is known, then, as was shown in (4), this system can be reduced to standard form. After this, the averaging procedure under consideration can be applied to the resulting system.

3. Let a system of nonlinear integral equations of Volterra type be given:

$$u(t) = \varepsilon \int_0^t \Phi(t, s, u(s)) ds. \quad (7)$$

Differentiating (7), we find

$$\dot{u} = \varepsilon \Phi(t, t, u) + \varepsilon \int_0^t \frac{\partial \Phi(t, s, u(s))}{\partial t} ds. \quad (8)$$

System (8) is integro-differential, and averaging Theorems 1 and 2 are applicable to it.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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