

THE SCHRÖDINGER EQUATION IN HILBERT SPACE

E. A. Begovatov

1970

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-197001.32408>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 517.954.4:517.937+513.88

MATHEMATICS

E. A. BEGOVATOV

THE SCHRÖDINGER EQUATION IN HILBERT SPACE

(Presented by Academician A. A. Dorodnitsyn, 8 X 1969)

Let H be a Hilbert space. Consider the set of all Bochner-integrable functions $f(x)$ ($-\infty < x < \infty$) with values in H . This set forms a new Hilbert space H_1 , if the inner product in it is defined by the equality

$$(f, g)_{H_1} = \int_{-\infty}^{\infty} (f(x), g(x))_H dx.$$

Consider in the space H_1 the differential operator

$$L\varphi = [d^2/dx^2 - Q(x)]\varphi, \quad (1)$$

defined on the set of functions from H_1 with values in D (here D is the domain of definition of the operator $Q(x)$) and having a strong second continuous derivative.

In the present note conditions are determined on a family of linear, generally speaking unbounded in H , operators $Q(x)$, under which the differential operator (1) (more precisely, its closure) is the generator of a strongly continuous contraction semigroup. It will also be shown that the resolvent of this operator is an integral operator with a certain kernel, which by analogy with the scalar case is called the operator Green's function.

We shall assume that:

α . The closed linear operator $Q(x)$, for each $x \in R$, has a dense in H , x -independent domain of definition D , and for any $\lambda \geq 0$ the operator $Q(x) + \lambda I$ has a bounded inverse. The operator $Q(x)$ is uniformly bounded below:

$$\operatorname{Re}(Q(x)\varphi, \varphi)_H \geq \delta(\varphi, \varphi)_H, \quad \varphi \in D.$$

The condition that the domain of definition D be independent of x , by virtue of the closed graph theorem, entails the boundedness in H of the operator

$Q(x) \cdot Q^{-1}(y)$, and the norm of this operator, generally speaking, is bounded by a constant depending on x and y . The following condition imposes a restriction on the growth of this constant.

β . There exist constants C and $k > 0$ such that

$$\|Q(x)Q^{-1}(y)\| \leq C \exp\{k|x-y|\}$$

and the operator $Q(x)Q^{-1}(y)$ for $|x-y| < 1$ satisfies the Hölder condition

$$\|[Q(x) - Q(y)]Q^{-1}(y)\| \leq C|x-y|^\varepsilon, \quad \varepsilon \in (0, 1);$$

moreover,

$$\|Q^{-1-\delta}(x)[Q(x) - Q(y)]\| \leq C|x-y|^\varepsilon, \quad \delta < \varepsilon,$$

or

$$\|[Q^{1/2}(x) - Q^{1/2}(y)]Q^{-1/2}(y)\| \leq C|x-y|^\varepsilon.$$

Let

$$Q_R = \frac{1}{2}(Q + Q^*) \quad \text{and} \quad Q_I = \frac{1}{2i}(Q - Q^*)$$

—the real and

imaginary components of the operator Q . We impose additional restrictions on these components.

γ . The operators Q_R and Q_I are semibounded on a set dense in H , and one of the three self-adjoint extensions in the Friedrichs sense, $\widetilde{Q}_R$, $\widetilde{Q}_I$, or $\overline{Q_R + Q_I}$, has a completely continuous resolvent, while the least eigenvalue $a_1(x)$ satisfies the condition of A. M. Molchanov

$$\lim_{|x| \rightarrow \infty} \int_x^{x+\omega} a_1(x) dx = \infty \quad \text{for any } \omega > 0.$$

Condition α makes it possible to construct the operator $\chi(x) = \{Q(x) + \mu^2 I\}^{1/2}$, whose resolvent set is the entire right half-plane ⁽¹⁾. In this case the operator χ generates a strongly continuous contraction semigroup $U_\mu^x(t) = \exp\{-t\chi(x)\}$, for which

$$\|\chi^\alpha \exp\{-t\chi\}\| \leq C t^{-\alpha} \exp\{\gamma \mu t\}, \quad \alpha \geq 0, \quad (2)$$

where $\gamma > 0$ and does not depend on $\mu > \mu_0$ and $t > 0$. Using, moreover, condition β and (2), one can show that

$$\|Q^{-\beta}(x)[Q^\alpha(x) - Q^\alpha(y)]\| \leq C|x - y|^\varepsilon \quad \text{for } \beta - \delta > \alpha; \quad (3)$$

$$\|Q^{1/2}(x)[U_\mu^x(t + \Delta t) - U_\mu^x(t)]\| \leq C\{\Delta t\}t^{-2} \exp\{-\gamma\mu t\}; \quad (4)$$

$$\|\chi^n(x)[U_\mu^x(t) - U_\mu^y(t)]\chi^{-k}(y)\| \leq C \frac{|x - y|^\varepsilon}{t^{n+\delta-k}} \exp\{-\gamma\mu t\}, \quad (5)$$

where $n = 0, 1, 2$; $k = 0, 1$; $n - k \leq 1$ and $\delta < \varepsilon$. From condition β and (2) it follows that for any $x \neq y \in R$

$$\|(Q(x) - Q(y))Q^{-1/2}(y)U_\mu^y(|x - y|)\| \leq \frac{C \exp\{-\gamma\mu|x - y|\}}{|x - y|^{1-\varepsilon}},$$

therefore the integral equation

$$G(x, y, \mu) = F(x, y, \mu) + \int_{-\infty}^{\infty} G(x, z, \mu)[Q(y) - Q(x)]F(z, y, \mu) dz, \\ F(x, y, \mu) = \frac{1}{2}\chi^{-1}(y)e^{-|x-y|\chi(y)} \quad (6)$$

has a unique solution $G(x, y, \mu)$, which is determined by the formula

$$G(x, y, \mu) = F(x, y, \mu) + \int_{-\infty}^{\infty} F(x, z, \mu)\Phi(z, y, \mu) dz, \quad (7)$$

where $\Phi(x, y, \mu)$ is an operator for which the following inequality holds: for any $\frac{\varepsilon}{1+\varepsilon} < \eta < \varepsilon$,

$$\|\Phi(x + \Delta x, y, \mu) - \Phi(x, y, \mu)\| \leq C|\Delta x|^{\varepsilon-\eta}|x - y|^{\varepsilon-1}, \quad |x - y| < 1.$$

Theorem 1. *If the positive number μ is sufficiently large and conditions α and β are satisfied, then equation (6) has a unique solution $G(x, y, \mu)$, which is the Green's function for the equation*

$$[L - \mu^2 I]\varphi = -f,$$

i.e., for any function $f \in H_1$ satisfying the Hölder condition, the integral

$$g = Af = \int_{-\infty}^{\infty} G(x, y, \mu) f(y) dy \quad (8)$$

gives a solution of equation (7).

The proof of the theorem is based on the method of P. E. Sobolevskii ⁽²⁾, and results ⁽¹⁾ are used here. In proving the existen-

of the second continuous strongly differentiable with respect to x operator function $G(x, y, \mu)$, the uniform boundedness with respect to x and the continuity of the operator

$$\int_{-\infty}^{\infty} Q(x) F(x, y, \mu) dy,$$

which follow from inequalities (2), (3), and (5).

Since the set of functions $f(x)$ satisfying the Hölder condition is dense in H_1 , it follows from the theorem of B. M. Lidskii ((3), p. 110) and B. M. Levitan and T. A. Suvorchenkova ((4), p. 56) that the operator (8) is completely continuous in H_1 , if the family of operators $Q(x)$ satisfies, in addition, condition γ . From condition α it is easy to obtain that in H_1 the inequality

$$0 \leq \operatorname{Re}(Af, f)_{H_1} \leq \frac{(f, f)_{H_1}}{\delta + \mu}$$

holds.

Therefore, by what was said above, μ is a regular point of the operator A ((5), p. 302), and the integral equation

$$G(x, y) = G(x, y, \mu) + \int_{-\infty}^{\infty} G(x, z, \mu) G(z, y) dz \quad (9)$$

has a unique solution.

Splitting the integral operator in (9) into a finite-dimensional operator and an integral operator whose norm is sufficiently small, one can show that the operator $G(x, y)$ is uniformly bounded in H . By virtue of the properties of the operator Green' s function $G(x, y, \mu)$, one can prove the following theorem.

Theorem 2. *If conditions α, β, γ are satisfied, then there exists a unique Green' s function $G(x, y)$ for the problem $L\varphi = -f$.*

The extension by continuity of the integral operator generated by the operator Green' s function $G(x, y)$ to all of H_1 leads to a certain closed extension of the operator L , whose range is all of H_1 . If one uses condition α and the Hille-Yosida theorem ((1), p. 109), one can obtain

Corollary 1. *Under the assumptions of Theorem 2, the closure of the operator L is the infinitesimal generator of a strongly continuous contraction semigroup $T_t = \exp\{-t\bar{L}\}$; moreover,*

$$dT_t v/dt = d^2 T_t v/dx^2 - Q(x)T_t v$$

for any $v \in D(L)$ and $t > 0$.

If the semigroups generated by the operators d^2/dx^2 and $Q(x)$ are denoted by R_t and Q_t , respectively, then, applying Trotter's theorem (6), one can obtain a representation of the semigroup T_t in terms of R_t and Q_t .

Corollary 2. *Under the assumptions of Theorem 2, the equality*

$$T_t v = \lim (R_h Q_h)^{[t/h]} v$$

holds for any $v \in H_1$.

Remark 1. If one uses the result of article (7), condition γ can be somewhat weakened.

Remark 2. In the case of a self-adjoint operator $Q(x)$, results analogous to Theorem 1 were obtained in (8), and in the case of a finite interval results analogous to Theorems 1 and 2 were obtained in (9).

In conclusion, I take the opportunity to express my gratitude to B. M. Levitan, conversations with whom substantially contributed to the writing of this paper.

Kazan State University
named after V. I. Ulyanov-Lenin

Received
3 X 1969

References

1. S. G. Krein, *Linear differential equations in Banach spaces*, Moscow, 1967.
2. P. E. Sobolevskii, *Trudy Moskov. Mat. Obshch.*, **10**, 297 (1961).
3. B. M. Lidskii, *Trudy Moskov. Mat. Obshch.*, **8**, 84 (1959).
4. B. M. Levitan, T. A. Suvorchenkova, *Functional analysis and its applications*, **2**, No. 2 (1968).
5. M. Sh. Gokhberg, M. G. Krein, *Introduction to the theory of nonselfadjoint operators*, Moscow, 1965.

6. H. F. Trotter, Proc. Am. Math. Soc., **10**, No. 4 (1959).
7. V. P. Maslov, *Functional analysis and its applications*, **2**, No. 2 (1968).
8. B. M. Levitan, Mat. Sbornik, **76** (118), 2 (1968).
9. G. I. Laptev, Litovsk. Mat. Sbornik, **8**, No. 1 (1968).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.