

A PROBLEM WITH TWO FREE BOUNDARIES FOR AN ELLIPTIC EQUATION AND OPTIMAL STOPPING OF A MARKOV PROCESS

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Abstract

Full Text

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MATHEMATICS

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A PROBLEM WITH TWO FREE BOUNDARIES FOR AN ELLIPTIC EQUATION AND OPTIMAL STOPPING OF A MARKOV PROCESS

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1. Let E_n be n -dimensional Euclidean space, U a bounded domain in E_n with twice continuously differentiable boundary,

$$Lu(x) = a_{ij}(x)u_{ij}(x) + b_i(x)u_i(x) - c(x)u(x)$$

an elliptic differential operator acting on functions $u \in W_p^2(U)$ (see ⁽¹⁾). Here u_{ij} is the mixed derivative with respect to x_i, x_j ; u_i is the derivative with respect to x_i ; summation over the indices i, j is assumed. Suppose that a_{ij}, b_i, c are bounded, $c \geq 0$, the matrix a_{ij} is uniformly nondegenerate, and, for $n \leq 2$, $p = 2$, while for $n > 2$ the $a_{ij}(x)$ are continuous in E_n , $p > n/2$. Let two functions $\psi_1, \psi_2 \in W_p^2(U)$ also be given such that $\psi_1 \geq \psi_2$ in U , $\psi_1 = \psi_2$ on ∂U .

Theorem 1. There exists, and moreover is unique, a function $u \in W_p^2(U)$ possessing the following two properties: a) $\psi_1(x) \geq u(x) \geq \psi_2(x)$ for $x \in U \cup \partial U$; b) $(-1)^i L u \geq 0$ (a.e.) on

$$\Gamma_i = \{x : (-1)^i [u(x) - \psi_i(x)] > 0\}$$

for $i = 1, 2$. In addition, if $p > n$, then $\text{grad}(u - \psi_i) = 0$ at all points of the set $U \setminus \Gamma_i$, $i = 1, 2$.

The problem consisting in finding a function satisfying conditions a), b) is called a problem with two free boundaries, because when $\psi_1 > \psi_2$ in U it is equivalent to the problem of finding a function $u \in W_p^2(U)$ and two closed sets G_1, G_2 such that a) is satisfied, $u = \psi_1$ and $Lu \geq 0$ (a.e.) on G_1 , $u = \psi_2$ and $Lu \leq 0$ (a.e.) on G_2 , and

$$Lu = 0$$

(a.e.) on $U \setminus (G_1 \cup G_2)$.

This problem is a generalization of the problem with one free boundary and is directly related to problems with a variational inequality ⁽²⁾. Namely, if $L\psi_1 < 0$, then from Theorem 1 it is not difficult to derive that u is the unique function in $W_p^2(U)$ such that

$$u \geq \psi_2, \quad (u - \psi_2)|_{\partial U} = 0,$$

$$Lu \leq 0$$

(a.e.) on $\{x : u(x) = \psi_2(x)\}$, and

$$Lu = 0$$

(a.e.) on $\{x : u(x) > \psi_2(x)\}$.

All these results can be obtained with the aid of certain facts from the theory of Markov processes. They are presented below.

2. Let X be a standard process in a locally compact space E (see ⁽³⁾), B the space of Borel bounded functions f with norm

$$\|f\| = \sup\{|f| : x \in E\},$$

R_λ the resolvent of the process X , m some probabilistic regular measure on E , $p > 1$. Suppose that there exists a constant N such that for all $x \in E$, $f \in B$ the function $R_0 f(x)$ is continuous and

$$R_0 |f|(x) \leq N \|f\|_p,$$

where $\|f\|_p$ is the norm in the space L_p of functions integrable to the p -th power with respect to the measure m . Let two Borel functions $\psi_1(x), \psi_2(x)$ also be given, with $\psi_1 \geq \psi_2$. We want to study the function

$$v(x) = \inf_{\tau} \sup_{\sigma} M_x \{ \psi_1(x_\tau) \chi_{\tau < \sigma} + \psi_2(x_\sigma) \chi_{\sigma \leq \tau} \},$$

where the lower and upper bounds are taken over the set of all Markov times τ and σ , respectively.

The problem of studying $v(x)$ is a generalization of the problems considered in (4–6). Our method differs from the methods known from (4–6); it consists in representing the function $v(x)$ in the form $R_0 f(x)$, where $f \in L_p$. In proving this representation we use the method of “continuous” stopping. The main technical tool is the following lemma.

For $c \in B$ put

$$R_c f(x) = M_x \int_0^\infty \exp \left[- \int_0^t c(x_s) ds \right] f(x_t) dt,$$

and, for $n, m > 0$, let

$$u_{nm} = \inf \sup R_{c_1+c_2} (c_1 \psi_1 + c_2 \psi_2),$$

where the $\inf(\sup)$ is taken over the set of all functions $c_1(c_2)$ such that $0 \leq c_1 \leq n$ ($0 \leq c_2 \leq m$),

$$\delta_{nm} = -(u_{nm} - \psi_1)_+, \quad \rho_{nm} = (u_{nm} - \psi_2)_-^*.$$

Lemma. Suppose that, for some $f^i \in L_p$, $\psi_i = R_0 f^i$ ($i = 1, 2$). Then:

a) $u_{nm} = R_c (n \delta_{nm} + m \rho_{nm})$;

b) if $n > m$, then

$$|\delta_{nm}| \leq R_n f_-^1, \quad \rho_{nm} \leq R_m n R_n f_-^1 + R_m f_+^2;$$

c)

$$\begin{aligned} u_{nm}(x) &= M_x \left[\int_0^{\tau \wedge \sigma} (n \delta_{nm} + m \rho_{nm})(x_t) dt + u_{nm}(x_{\tau \wedge \sigma}) \right] \\ &= \sup_\sigma M_x \left[\int_0^{\tau \wedge \sigma} n \delta_{nm}(x_t) dt - \rho_{nm}(x_\sigma) \chi_{\sigma \leq \tau} + \psi_2(x_\sigma) \chi_{\sigma \leq \tau} + u_{nm}(x_\tau) \chi_{\tau < \sigma} \right] \\ &= \inf_\tau \sup_\sigma M_x [\psi_1(x_\tau) \chi_{\tau < \sigma} + \psi_2(x_\sigma) \chi_{\sigma \leq \tau} - \delta_{nm}(x_\tau) \chi_{\tau < \sigma} - \rho_{nm}(x_\sigma) \chi_{\sigma \leq \tau}]. \end{aligned}$$

Proof. Consider the operator

$$F_{nm} v = R_{n+m} [(n+m)v - n(v - \psi_1)_+ + m(v - \psi_2)_-]$$

in the space B . From the inequality

$$|(n+m)t_1 - n(t_1 - a)_+ + m(t_1 - b)_-| - |(n+m)t_2 - n(t_2 - a)_+ + m(t_2 - b)_-| \leq (n+m)|t_1 - t_2|$$

it follows that

$$|F_{nm} v_1 - F_{nm} v_2| \leq (n+m) R_{n+m} |v_1 - v_2|.$$

Further, note that from the estimate $|R_0 f| \leq N \|f\|$ it follows that

$$(n+m) \|R_{n+m} f\| \leq \varepsilon_{n+m} \|f\|,$$

where $\varepsilon_{n+m} < 1$ and does not depend on f .

Thus the operator F_{nm} is a contraction in B . By Banach's theorem there exists $v_{nm} \in B$ such that

$$v_{nm} = F_{nm} v_{nm}.$$

Next, for $0 \leq d \leq k$ the equality

$$R_d f = \sum_{i=0}^{\infty} [R_k(k-d)]^i R_k f$$

holds. Hence, for

$$d = c_1 + c_2, \quad k = n + m, \quad f = dv_{nm} - n(v_{nm} - \psi_1)_+ + m(v_{nm} - \psi_2)_-,$$

in view of the equality

$$R_{n+m} f = c_{nm} + R_{n+m}(d-k)v_{nm},$$

it easily follows that

$$R_{c_1+c_2} f = v_{nm}$$

for $0 \leq c_1 \leq n$, $0 \leq c_2 \leq m$.

From this equality and the inequality

$$c_2(v_{nm} - \psi_2) + m(v_{nm} - \psi_2)_- \geq 0$$

we obtain

$$v_{nm} = \sup R_{c_1+c_2} [c_1 v_{nm} - n(v_{nm} - \psi_1)_+ + c_2 \psi_2],$$

and, as a consequence of the inequality

$$c_1(v_{nm} - \psi_1) - n(v_{nm} - \psi_1)_+ \leq 0,$$

this gives

$$v_{nm} = \inf \sup R_{c_1+c_2} (c_1 \psi_1 + c_2 \psi_2),$$

i.e. $v_{nm} = u_{nm}$. Assertion a) now follows from the equality

$$R_{c_1+c_2} f = v_{nm}$$

for $c_1 = c_2 = 0$.

Let us again turn to the equality

$$u_{nm} = R_{c_1+c_2} [(c_1 + c_2)u_{nm} + n\delta_{nm} + m\rho_{nm}].$$

For $n > m$, $a \geq b$ we have

$$nt - n(t - a)_+ + m(t - b)_- \leq na,$$

therefore, for $c_1 = n$, $c_2 = 0$, $n > m$, we obtain

$$u_{nm} \leq nR_n\psi_1,$$

and since

$$nR_n\psi_1 = nR_nR_0f^1 = R_0f^1 - R_nf^1 = \psi_1 - R_nf^1,$$

it follows that

$$u_{nm} \leq \psi_1 - R_nf^1 \leq \psi_1 + R_nf_-^1$$

and

$$|\delta_{nm}| \leq R_nf_-^1.$$

If, however, we take $c_1 = 0$, $c_2 = m$, then

$$u_{nm} \geq -R_m nR_nf_-^1 + R_m[mu_{nm} + m\rho_{nm}] \geq -R_m nR_nf_-^1 + mR_m\psi_2,$$

and

$$\rho_{nm} \leq R_m nR_nf_-^1 + R_m f_+^2.$$

Assertion b) is proved.

The first equality in c) follows from a) and the strong Markov property of X . The second equality is valid because, on the one hand,

$$u_{nm} \geq \psi_2 - \rho_{nm}$$

and $\rho_{nm} \geq 0$, while, on the other hand, if

$$\sigma = \inf\{t : \rho_{nm}(x_t) > 0\},$$

then

$$\int_0^{\tau \wedge \sigma} \rho_{nm}(x_t) dt = 0$$

and

$$\psi_2(x_\sigma) - \rho_{nm}(x_\sigma) = u_{nm}(x_\sigma).$$

The third equality is obtained by analogous arguments from the second. The lemma is proved.

From this lemma the following theorems are derived rather simply. In their statements,

$$M_x(\tau, \sigma) = M_x[\psi_1(x_\tau)\chi_{\tau < \sigma} + \psi_2(x_\sigma)\chi_{\sigma \leq \tau}].$$

Theorem 2. Suppose that, for some $f^i \in B$, $\psi_i = R_2 f^i$ ($i = 1, 2$); then there exists a function $f \in B$ such that $v = R_0 f$, and hence v is continuous. Further,

$$(-1)^i f \leq 0 \quad \text{on} \quad \Gamma_i(v) = \{x : (-1)^i[v(x) - \psi_i(x)] > 0\},$$

$$\lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \|v - u_{nm}\| = 0, \quad v(x) = \sup_{\sigma} M_x(\tau_1, \sigma) = \inf_{\tau} M_x(\tau, \tau_2),$$

$$* \quad a_+ = \frac{1}{2}(|a| + a), \quad a_- = \frac{1}{2}(|a| - a).$$

where $\tau_i = \inf\{t : v(x_t) = \psi_i(x_t)\}$ ($i = 1, 2$). Moreover, if $u, h \in B$ are functions related by the relation $u = R_0 h$, $\psi_1 \geq u \geq \psi_2$, and $(-1)^i h \leq 0$ on $\Gamma_i(u)$, for $i = 1, 2$, then $u = v$.

Theorem 3. Suppose there exists a constant N_1 such that $\lambda \|R_\lambda f\|_p \leq N_1 \|f\|_p$ for all $f \in B$, $\lambda > 0$. Then in the formulation of Theorem 2 one may replace B by L_p . Moreover, as τ_i one may also take

$$\inf\{t : v(x_t) = \psi_i(x_t), \quad x_t \in \text{supp}[(-1)^i f]^+\}.$$

Before formulating Theorem 4, take two closed sets G_1, G_2 and denote by $\mathfrak{M}_i$ the set of Markov times such that

$$P_x\{x_\tau \in G_i\} = P_x\{\tau < \zeta\}$$

for all x , and by $\mathfrak{N}_i$ the set of hitting times of closed subsets of G_i .

Theorem 4. Suppose the assumption of Theorem 3 is satisfied and there exist sequences of functions $f_n^i \in L_p$ such that

$$\text{supp}[(-1)^i f_n^i]^+ \subset G_i, \quad R_0 f_n^1 \geq R_0 f_n^2, \quad \lim_{n \rightarrow \infty} \|[\psi_i - R_0 f_n^i] \chi_{G_i}\| = 0.$$

Then

$$\inf_{\tau \in \mathfrak{M}_1} \sup_{\sigma \in \mathfrak{M}_2} M_x(\tau, \sigma) = \inf_{\tau \in \mathfrak{M}_1} \sup_{\sigma \in \mathfrak{M}_2} M_x(\tilde{\tau}, \sigma) = \inf_{\tau \in \mathfrak{M}_1} \sup_{\sigma \in \mathfrak{N}_2^x} M_x(\tau, \sigma).$$

Moreover, all expressions in the last equality are continuous.

We make some remarks concerning the proofs of Theorems 1-4. Theorem 1 is derived from Theorem 3 with the aid of the known properties of $R_0 f$ for quasidiffusion processes; Theorem 4 is also easily derived from Theorem 3. The proofs of Theorems 2 and 3 follow one and the same plan. First, with the aid of Lemma b), it is proved that

$$\lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \|\rho_{nm} + |\delta_{nm}|\| = 0;$$

then, using Lemma c), that $\|u_{nm} - v\| \rightarrow 0$. After this, from the estimate $R_0|f| \leq N\|f\|_p$ and Lemmas a), b), one obtains the equality $v = R_0 f$ and the sign properties of $(-1)^i f$. The equality

$$v(x) = \sup_{\sigma} M_x(\tau_1, \sigma)$$

follows from the equality of u_{nm} to the third term in equality c) of the lemma and from the estimates $|\delta_{nm}| \leq R_n f^-$, $R_0|f| \leq N\|f\|_p$, $\lambda\|R_\lambda f\|_p \leq N_1\|f\|_p$. The last assertion of Theorem 2 is almost obvious.

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CITED LITERATURE

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