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Abstract

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MATHEMATICS

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ON THE UNIQUENESS OF THE SOLUTION OF THE CAUCHY PROBLEM FOR LINEAR EVOLUTION SYSTEMS WITH VARIABLE COEFFICIENTS

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In the notes ^(1,2) results were set forth on the study of solutions of linear evolution systems of arbitrary order with variable coefficients, the spatial part of which is a Petrovskii-elliptic system of order m , while differentiation with respect to the time coordinate t has maximal order less than m . It was assumed about the solutions that their components not contained in a certain cone K of the N -dimensional complex space C^N have uniformly bounded L_1 -norm. For such solutions inequalities of the type of L_1 -Carleman inequalities were established, from which assertions were derived about the character of growth of solutions in the simplest domains. From these assertions there followed refinements and generalizations of the classical uniqueness theorems for the solution of the Cauchy problem of Widder and Täcklind.

A more detailed analysis of the means by which such theorems are proved has led to the establishment of entirely new uniqueness theorems for the solution of the Cauchy problem for **arbitrary** linear evolution systems with variable coefficients, solved with respect to derivatives in the time coordinate t . It is assumed that the solution lies in a cone and satisfies a certain L_1 -estimate. The proof is based on a special lemma (Lemma 1) on the possibility of an L_1 -estimate with weight in a layer of small height for the part of the solution lying in the cone. For solutions of systems with elliptic spatial part the additionally needed L_1 -estimate is obtained automatically.

For systems with constant coefficients having an elliptic spatial part, and for Petrovskii-parabolic systems with variable growing coefficients, quite complete and, in a certain sense, final results have been obtained. For Petrovskii-parabolic equations with weight of the form $2(2\nu + 1)$, ν an integer, a theorem is proved on the coincidence of two solutions under restrictions on their components not belonging to the cone.

The present article is devoted to the exposition of these results. The notation

and definitions are the same as in (2).

1. Arbitrary linear evolution systems

Consider the system

$$\begin{aligned}\mathcal{L}\left(t, x; \frac{\partial}{\partial t}, D_x\right) u &\equiv -\frac{\partial u}{\partial t} + P(t, x; D_x)u \equiv \\ &\equiv -\frac{\partial u}{\partial t} + \sum_{|k| \leq m} A_k(t, x) D_x^k u = 0\end{aligned}\quad (1)$$

without any assumptions on the type of the matrix P .

We assume fulfilled

Condition α . $m > 1$, and the coefficients of the operator $P^*(t, x; D_x)$, Lagrange-adjoint to $P(t, x; D_x)$, have the following property: $A_k^*(t, x)|x|^{(|k|-m)/(m-1)}$ are bounded by a constant A in the layer $\Pi(0, \tau) = \Pi_\tau$ (for $|x| \leq 1$ the power factor is absent).

Theorem 1. Let a solution $u(t, x)$ of system (1) (satisfying condition a) in Π_T be continuous in $\bar{\Pi}_T$ and satisfy the conditions: 1) $u \in K$, where K is a cone in the complex space C^N , whose closure has only one common point (the origin) with some half-space;

2)

$$\|u(t, x)\|_{\Pi_{(0,T)}^{2,\infty}} \leq C \exp\{c|x^0|^{m'}\}, \quad m' = \frac{m}{m-1}, \quad \Pi_{(0,T)}^{2,\infty} \subset \Pi_T;$$

3) $u|_{t=T} = 0$. Then $u(t, x)$ is identically equal to zero in Π_T .

Theorem 1 follows from the following fundamental lemma.

Lemma 1. Let the coefficients of system (1) satisfy condition a, and let the solution $u(t, x)$ satisfy condition 2) of Theorem 1. Then there exist positive constants Q and t_0 , depending only on A , such that

$$\begin{aligned}&\iint_{\Pi_{t_0}} |u^+(t, x)| \exp\left\{-\frac{2c|x|^{m'}}{(1+Qt)^{1/(m-1)}}\right\} dt dx \leq \\ &\leq K(A) \left(\iint_{\Pi_{t_0}} |u^-(t, x)| \exp\left\{-\frac{2c|x|^{m'}}{(1+Qt)^{1/(m-1)}}\right\} dt dx + \right. \\ &\quad \left. + \int_{\Sigma^\infty} |u(t_0, x)| \exp\left\{-\frac{2c|x|^{m'}}{(1+Qt_0)^{1/(m-1)}}\right\} dx \right).\end{aligned}\quad (2)$$

The proof of Lemma 1 is based on the fact that for the auxiliary function

$$W(t, x) = t \exp \left\{ -\frac{2c\varphi(x)}{(1 + Qt)^{1/(m-1)}} \right\} e,$$

where $\varphi(x)$ is a smooth function equal to $|x|^{m'}$ for $|x| \geq 1$, and e is a vector determined by the cone,

$$\begin{aligned} \mathcal{L}^* w = & \left\{ 1 + t|x|^{m'} \left[\frac{2c}{m-1} \frac{Q}{(1 + Qt)^{m'}} + \sum_{|k| \leq m} (-2c)^{|k|} m'^{|k|} A_k^*(t, x) \times \right. \right. \\ & \left. \left. \times \frac{|x|^{(|k|-m)/(m-1)}}{(1 + Qt)^{|k|/(m-1)}} \left(\frac{x}{|x|} \right)^k + \frac{1}{|x|} \Psi(t, x) \right] \right\} \exp \left\{ -2c \frac{|x|^{m'}}{(1 + Qt)^{1/(m-1)}} \right\} e. \end{aligned}$$

Here $\Psi(t, x)$ is a bounded function; then, choosing Q sufficiently large and t_0 sufficiently small, we obtain that for $t \in [0, t_0]$:

$$\operatorname{Re}(\mathcal{L}^* w \cdot u^+) \geq \gamma_1 \exp \left\{ -2c \frac{|x|^{m'}}{(1 + Qt)^{1/(m-1)}} \right\} |u^+|; \quad (3)$$

Lemma 1 follows from Green's formula and inequality (3).

If $u^- = 0$ and $u|_{t=T} = 0$, then, successively applying Lemma 1 to the layers $\Pi_{(T-t_0, T)}$, $\Pi_{(T-2t_0, T-t_0)}$, and so on, we obtain the assertion of Theorem 1.

From Theorem 1 and Theorem 1 (inequality 6) from (2) it follows:

Theorem 2. Let the coefficients of system (1) satisfy condition a, and let $P(t, x; D_x)$ be a uniformly elliptic operator. If $u(t, x)$ is a solution of system (1) in Π_T , continuous in $\bar{\Pi}_T$, then from $u \in K$, $u|_{t=T} = 0$ it follows that $u(t, x) \equiv 0$ in Π_T .

2. Systems with constant coefficients. Here a system with constant coefficients is considered:

$$\mathcal{L}u \equiv -\frac{\partial u}{\partial t} + \sum_{|k| \leq m} A_k D_x^{ku} \equiv -\frac{\partial u}{\partial t} + P(D_x)u = 0, \quad (4)$$

where $P(D_x)$ is an elliptic operator in the sense of Petrovskii, $m > 1$.

Theorem 3. Let $u(t, x)$ be a weak solution in Π_T of system (4), continuous in $\bar{\Pi}_T$, and satisfying the condition

$$\|u^-(t, \xi)\|_{\Sigma_2^*} \leq C \exp[|x|^{m'} h(|x|)], \quad (5)$$

where $h(r)$ is a slowly increasing function ⁽³⁾ such that

$$\int_1^{\infty} \frac{dr}{r[h(r)]^{1/(m-1)}} = \infty. \quad (6)$$

If $u(0, x) = 0$, then $u(t, x) \equiv 0$ in Π_T .

The proof of the theorem is carried out with the aid of the method of N. N. Chaus ⁽³⁾; here the L_1 -hypoellipticity of first-order systems in t (with variable coefficients of arbitrary type) is essentially used, i.e., the possibility of obtaining an L_1 -estimate for solutions on sections $t = \text{const}$ from an L_1 -estimate on some $(n+1)$ -dimensional parallelepiped, and the possibility, in the method of N. N. Chaus, of replacing pointwise estimates by L_1 -estimates of solutions on sections $t = \text{const}$.

3. Parabolic systems in the sense of Petrovsky with coefficients growing with the growth of the spatial coordinates. Consider the system, uniformly parabolic in the sense of Petrovsky,

$$\mathcal{L}\left(t, x; \frac{\partial}{\partial t}, D_x\right)u \equiv \sum_{k_0 p + |k| \leq m} A_{k_0 k}(t, x) \frac{\partial^{k_0}}{\partial t^{k_0}} D_x^k u = 0. \quad (7)$$

Using the uniqueness theorem for the solution of the Cauchy problem established by V. S. Ryzh in ⁽⁴⁾, and estimate (6) of Theorem 1 ⁽²⁾, which is also valid for solutions of systems with coefficients growing in a certain way with the growth of the spatial coordinates, it is easy to prove the following assertion:

Theorem 4. Let system (7) be uniformly parabolic in the sense of Petrovsky, and let the operator $\mathcal{L}\left(t, x; \frac{\partial}{\partial t}, D_x\right)$ adjoint to $\mathcal{L}\left(t, x; \frac{\partial}{\partial t}, D_x\right)$ have coefficients $A_{k_0 k}^*(t, x)$ possessing the property: $A_{k_0 k}^*(t, x) \times |x|^{(k_0 p + |k| - m)/(p-1) - \varepsilon}$, $\varepsilon > 0$, are uniformly continuous and bounded in the layer Π_T (for $|x| < 1$ the power factor need not be written for $A_{k_0 k}^*(t, x)$). If $u(t, x)$ is a solution in Π_T of system (7), continuous in $\bar{\Pi}_T$, for which $\|u^{-1}(t, \xi)\|_{\Sigma_2^x} \leq C \exp\{|x|^{m'} h(|x|)\}$, where $h(r)$ is a slowly increasing function,

$$\int_1^{\infty} \frac{dr}{r h^{1/(m-1)}(r)} = \infty,$$

and $u(0, x) = 0$, then $u(t, x) \equiv 0$ in Π_T .

4. The theorem on the coincidence of solutions with identical initial conditions. Up to now a solution equal to zero for $t = 0$ has been considered, and, under certain restrictions on the possible growth of its component that does

not fall into the cone K , its identical equality to zero has been established. If, however, one considers two solutions $u_1(t, x)$, $u_2(t, x)$ coinciding for $t = 0$, and defines the function $u(t, x) = u_1(t, x) - u_2(t, x)$, then the conditions on $u_1^-(t, x)$, $u_2^-(t, x)$ give no information about $u^-(t, x)$, and therefore the theorems stated above are inapplicable. However, one can single out a class of equations for whose solutions, from restrictions on $u_1^-(t, x)$, $u_2^-(t, x)$, one obtains L_1 -estimates of $u_1(t, x)$, $u_2(t, x)$ without additional assumptions about their behavior on the initial hyperplane.

Consider the equation

$$\mathcal{L}u \equiv -\frac{\partial u}{\partial t} + \sum_{|k| \leq m} A_k(t, x) D_x^k u = 0. \quad (8)$$

Let the following condition be satisfied:

β . The range of values of the polynomial

$$P_0(t, x; \sigma) = \sum_{|k|=m} A_k(t, x) \sigma^k, \quad m > 1,$$

where σ is any real vector, lies in a sector of opening less than π of the right half-plane of the complex z -plane, i.e.,

$$|\arg P_0(t, x; \sigma)| \leq \varphi_1 < \pi/2.$$

With the angle φ_1 we define the cone (angle) K_{φ_2} ,

$$|\arg u - \varphi_0| \leq \pi/2 - \varphi_1 - \varepsilon_1 = \varphi_2, \quad \varepsilon_1 > 0.$$

In the cone K_{φ_2} the quantities u^+ , u^- are defined.

Theorem 5. Suppose that equation (8) satisfies the conditions β , and that the coefficients $\mathcal{L}^*$ satisfy the conditions of Theorem 4. If $u_1(t, x)$, $u_2(t, x)$ are solutions of equation (8) in Π_T , for which

$$\|u_i^-(t, \xi)\|_{\Sigma_{\xi}^x} \leq C_1 \exp\{|x|^{m'}/h(|x|)\}, \quad i = 1, 2,$$

$h(r)$ is a slowly increasing function such that

$$\int_1^\infty \frac{dr}{r(h(r))^{1/(m-1)}} = \infty,$$

then from the fact that $u_1(0, x) = u_2(0, x)$ it follows that

$$u_1(t, x) \equiv u_2(t, x) \quad \text{in } \Pi_T.$$

The proof is based on the possibility, under fulfillment of condition β , of constructing an auxiliary function of the form $tv(t, x)$, equal to zero on the boundary of the spherical half-neighborhood G , and such that in G

$$\operatorname{Re}(\mathcal{L}^*(tv(t, x)) \cdot u^+) \geq 0$$

and in the part of G adjacent to the hyperplane $t = 0$

$$\operatorname{Re}(\mathcal{L}^*(tv(t, x)) \cdot u^+) \geq \gamma|u^+|,$$

where γ is a positive constant. This makes it possible to obtain an estimate for the growth of solutions without conditions on the initial hyperplane.

Then one uses the fact that, for $m = 2(2\nu + 1)$, $\nu = 0, 1, \dots$, equation (8) is parabolic in the sense of Petrovskii, while for $m = 4\nu$ it is backward parabolic, and therefore Theorem 4 is applicable.

5. Estimate of solutions of a weighted homogeneous system in a half-space. In conclusion we present some consequences, obtained incidentally from Lemma 1, concerning the growth of solutions defined in a half-space. An immediate consequence of Lemma 1 is:

Lemma 2. *Suppose that the coefficients of system (1) satisfy condition α . If:*
1) $u(t, x)$ is a solution of system (1) in the half-space $t < 0$;

$$2) \quad \|u^-\|_{\Pi_{(-1,0)}^{1,x}} \leq C_1(1 + |x|)^{s_1}; \quad 3) \quad \|u(0, \xi)\|_{\Sigma_\xi^x} \leq C_2(1 + |x|)^{s_2};$$

$$4) \quad \|u\|_{\Pi_{(-1,0)}^{1,x}} \leq C \exp[c|x|^{m'}],$$

then there exist a constant $d < 1$, depending on A , and $\lambda(A, s_1, s_2)$, such that

$$\|u\|_{\mathcal{E}_d} \leq \lambda(A, s_1, s_2)(C_1 + C_2).$$

Here

$$\mathcal{E}_d = \{(\tau, \xi), d(\tau, \xi) < d\}, \quad d(t, x) = (|t|^{2/m} + |x|^2)^{1/2}.$$

From Lemma 2 it follows:

Theorem 6. *Suppose: 1) $u(t, x)$ is a solution of the weighted-homogeneous system*

$$\mathcal{L}_0 u \equiv -\frac{\partial u}{\partial t} + \sum_{|k|=m} A_k(t, x) D_x^k u = 0 \quad \text{in the half-space } t < 0;$$

$$2) \quad \|u^-\|_{\mathcal{E}_{d(t,x)}} \leq C_1 d(t, x)^{s_1}, \quad d(t, x) \geq 1; \quad 3) \quad \|u(0, \xi)\|_{\Sigma_{|x|}} \leq C_2 |x|^{s_2},$$

$$\|x\| \geq 1; \quad 4) \quad \|u\|_{\Pi_{(-T,0)}^{1,x}} \leq M(T, \eta) \exp\{\eta|x|^{m'}\}$$

for arbitrary positive T, η (if

$$\sum_{|k|=m} A_k(t, x) \sigma^k$$

is a uniformly elliptic matrix in the sense of Petrovskii, then 4) follows from 2), 3)), then there exists a constant $C_3 = C_3(A, s_1, s_2)$ such that

$$\|u\|_{\mathcal{E}_{d(t,x)}} \leq C_3 d(t, x)^{s_3}, \quad s_3 = \min(s_1, s_2 - m).$$

From the last theorem follows a sharpened Liouville theorem for systems with constant coefficients.

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Note: Figure translations are in progress. See original paper for figures.

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