

ON THE APPROXIMATION OF POLYNOMIALS OF TWO VARIABLES BY FUNCTIONS OF THE FORM

$$\varphi(x) + \psi(y)$$

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Abstract

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MATHEMATICS

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ON THE APPROXIMATION OF POLYNOMIALS OF TWO VARIABLES BY FUNCTIONS OF THE FORM $\varphi(x) + \psi(y)$

(Presented by Academician I. N. Vekua, 30 I 1970)

In the present work, for a certain class of functions of two variables, a simple method is proposed for computing the values of the best approximation by functions of the form $\varphi(x) + \psi(y)$; a method is established for finding the best approximating function for polynomials of two variables, by means of which the best approximating function is found.

Let $f(x, y)$ be a function continuous on the rectangle $Q = [a, b; c, d]$, and

$$E[f, \varphi + \psi; Q]_C = E[f] = \inf_{\varphi + \psi} \|f - (\varphi + \psi)\|, \quad \|f\| = \sup_{x, y \in Q} |f|.$$

Denote by Π the class of functions $f(x, y)$ having the following property: for arbitrary $x'' \geq x', y'' \geq y'$ from Q , the inequality

$$f(x'', y'') + f(x', y') \geq f(x'', y') + f(x', y'')$$

holds.

Let Π_- be the class of functions $f(x, y)$ for which $(-f) \in \Pi$. It is not difficult to understand that if $f_{xy} \geq 0$, then $f \in \Pi$. Indeed, for arbitrary $x'' \geq x', y'' \geq y'$ we have

$$0 \leq \int_{x'}^{x''} \int_{y'}^{y''} f_{xy} dx dy = f(x'', y'') + f(x', y') - f(x'', y') - f(x', y'').$$

The converse, in general, is not true, since the function $f(x, y)$ need not be differentiable.

Similarly, if $f_{xy} \leq 0$, then $f \in \Pi_-$.

Theorem 1. For an arbitrary continuous function $f(x, y) \in \Pi$,

$$E[f, \varphi + \psi; Q]_C = \frac{1}{4}[f(b, d) + f(a, c) - f(a, d) - f(b, c)].$$

This result, which is a strengthening of a theorem of T. J. Rivlin and R. J. Sibner ⁽¹⁾ (the case $f_{xy} \geq 0$, $Q = [0, 1; 0, 1]$), also differs in the method of proof.

We also note the relatively complicated methods ⁽²⁻⁴⁾ for determining the value of the best approximation in the general case.

Corollary 1. For an arbitrary continuous function $f(x, y) \in \Pi_-$,

$$E[f, \varphi + \psi; Q]_C = -\frac{1}{4}[f(b, d) + f(a, c) - f(a, d) - f(b, c)].$$

Corollary 2. If the mixed derivative f_{xy} preserves its sign on the rectangle Q , then

$$E[f, \varphi + \psi; Q]_C = \frac{1}{4}|f(b, d) + f(a, c) - f(a, d) - f(b, c)|.$$

Corollary 3. Let $f = x^\alpha y^\beta g(x, y)$, where $\alpha, \beta \geq 0$ are real numbers and $Q = [0, b; 0, d]$.

Then:

- a) if $f \in \Pi$, then $E[f, \varphi + \psi, Q]_C = \frac{1}{4}f(b, d)$;
- b) if $f \in \Pi_-$, then $E[f, \varphi + \psi, Q]_C = -\frac{1}{4}f(b, d)$.

Corollary 4. Let $S = [0, 1; 0, 1]$. Then for arbitrary real $\alpha, \beta \geq 0$,

$$E[x^\alpha y^\beta, \varphi + \psi; S]_C = E[xy, \varphi + \psi; S]_C = \frac{1}{4}.$$

This follows immediately from Corollary 3 and from the fact that $x^\alpha y^\beta \in \Pi$. Consider the approximation of the polynomial in two variables

$$z = \sum_{p=0}^m \sum_{q=0}^n A_{pq} x^p y^q$$

on the rectangle Q by all possible functions of the form $\varphi(x) + \psi(y)$ in the metric of the space C . Taking into account

$$\sum_{p=0}^m \sum_{q=0}^n A_{pq} x^p y^q = \sum_{p=1}^m \sum_{q=1}^n A_{pq} x^p y^q + \sum_{p=0}^m A_{p0} x^p + \sum_{q=1}^n A_{0q} y^q,$$

we conclude that

$$E \left[\sum_{p=0}^m \sum_{q=0}^n A_{pq} x^p y^q, \varphi + \psi \right] = E \left[\sum_{p=1}^m \sum_{q=1}^n A_{pq} x^p y^q, \varphi + \psi \right],$$

and if $\varphi_0 + \psi_0$ is the best approximant to the polynomial $\sum_{p=0}^m \sum_{q=0}^n A_{pq} x^p y^q$, then

$$\varphi_0 + \psi_0 - \left(\sum_{p=0}^m A_{p0} x^p + \sum_{q=1}^n A_{0q} y^q \right)$$

is the best approximant to the polynomial $\sum_{p=1}^m \sum_{q=1}^n A_{pq} x^p y^q$.

Therefore, in what follows we shall assume

$$z = \sum_{p=1}^m \sum_{q=1}^n A_{pq} x^p y^q.$$

We propose the following method α for determining the best approximating function:

$$z = \sum_{p=1}^m \sum_{q=1}^n A_{pq} x^p y^q,$$

$$z_1 = z, \quad z_{2n} = z_{2n-1} - g_n, \quad z_{2n+1} = z_{2n} + h_n,$$

where

$$g_n = g_n(x) = \frac{1}{2} \left[\max_{c \leq y \leq d} z_{2n-1} + \min_{c \leq y \leq d} z_{2n-1} \right],$$

$$h_n = h_n(y) = \frac{1}{2} \left[\max_{a \leq x \leq b} z_{2n} + \min_{a \leq x \leq b} z_{2n} \right],$$

$$\widehat{\max} z = \sum_{p=1}^m \sum_{q=1}^n \max A_{pq} x^p y^q,$$

$$\check{\min} z = \sum_{p=1}^m \sum_{q=1}^n \min A_{pq} x^p y^q;$$

$\widehat{\max} z$ denotes the sum of the maxima of all terms of the polynomial, and $\check{\min} z$ the sum of the minima. Obviously,

$$\widehat{\max} z \geq \max z; \quad \check{\min} z \leq \min z.$$

Denote

$$\lim_{n \rightarrow \infty} g_n(x) = g_0(x) = g_0, \quad \lim_{n \rightarrow \infty} h_n(y) = h_0(y) = h_0.$$

We assert that

$$\|z - (g_0 - h_0)\| = E[z],$$

i.e. $g_0(x) + h_0(y)$ is a best approximating function.

We shall need some auxiliary facts.

Lemma 1. If, for nonnegative numbers c, d, y , for some fixed i ,

$$y^i \geq \frac{1}{2}(c^i + d^i), \quad y^{i+1} < \frac{1}{2}(c^{i+1} + d^{i+1}),$$

then

$$y^{i+k} < \frac{1}{2}(c^{i+k} + d^{i+k}), \quad k = 2, 3, \dots$$

Lemma 2. If, for nonnegative numbers c, d , and y ,

$$y < \frac{1}{2}(c + d),$$

then

$$y^i < \frac{1}{2}(c^i + d^i), \quad i = 2, 3, \dots$$

Corollary. If $y^n \geq \frac{1}{2}(c^n + d^n)$, then

$$y^i \geq \frac{1}{2}(c^i + d^i), \quad i = 1, 2, \dots, n-1.$$

Indeed, the existence of at least one inequality

$$y^{i_0} < \frac{1}{2}(c^{i_0} + d^{i_0}), \quad i_0 \leq n-1,$$

would imply, by one of Lemmas 1 and 2, that

$$y^n < \frac{1}{2}(c^n + d^n).$$

Theorem 2. Method α makes it possible to obtain, for the polynomial

$$z = \sum_{p=1}^m \sum_{q=1}^n A_{pq} x^p y^q, \quad A_{pq} \geq 0,$$

the best approximating function of the form $\varphi(x) + \psi(y)$ on the rectangle $[a, b; c, d]$, $a, c \geq 0$, and this function is the polynomial

$$\frac{1}{2} \sum_{p=1}^m \sum_{q=1}^n A_{pq} \left[x^p (c^q + d^q) + y^q (a^p + b^p) - \frac{1}{2} (a^p + b^p)(c^q + d^q) \right].$$

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CITED LITERATURE

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