

The Cauchy Problem for Weakly Hyperbolic Equations

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Abstract

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MATHEMATICS

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THE CAUCHY PROBLEM FOR NONSTRICTLY HYPERBOLIC EQUATIONS

(Presented by Academician S. L. Sobolev on 12 I 1970)

In the main, our notation corresponds to ⁽¹⁾. We consider the space R^{l+1} of vectors $x = (x_1, \dots, x_l)$; x_0 will sometimes be denoted by t . By S_t we denote the hyperplane $x_0 = t$. Further,

$$D_x^\beta = \partial^{|\beta|} / \partial x_0^{\beta_0} \cdots \partial x_l^{\beta_l}, \quad |\beta| = \beta_0 + \cdots + \beta_l.$$

Introduce the norms

$$|D^n f, S_t|^2 = c \sup_{|\beta| \leq n} |D_x^\beta f_0, S_t|_2^2 = c \sup_{|\beta| \leq n} \int_{S_t} |D_x^\beta f|^2 dx_1 \cdots dx_l,$$

$$\|D^n f, S_t\|^2 = c \sup_{|\beta| \leq n, K_t \subset S_t} |D_x^\beta f, K_t|_2^2 = c \sup_{|\beta| \leq n, K_t \subset S_t} \int_{K_t} |D_x^\beta f|^2 dx_1 \cdots dx_l,$$

where K_t is the unit cube in S_t .

Introduce the quasinorms

$$|D^{n,\infty} f, S_t, \rho| = \sum_{s=0}^{\infty} \frac{\rho^s}{s!} \sup_{|\sigma|=s} |D^n D_x^\sigma f, S_t|,$$

$$\|D^{n,\infty} f, S_t, \rho\| = \sum_{s=0}^{\infty} \frac{\rho^s}{s!} \sup_{|\sigma|=s} \|D^n D_x^\sigma f, S_t\|,$$

$$\sigma = (0, \sigma_1, \dots, \sigma_l).$$

In what follows we assume everywhere that $n > l/2$.

Let a formal series be given

$$\Phi(t, \rho) = \sum_{s=0}^{\infty} \frac{\rho^s}{s!} \Phi_s(t)$$

and let $\alpha > 1$. Introduce the operator λ

$$\lambda \Phi(t, \rho) = \sum_{s=0}^{\infty} \frac{\rho^s}{(s!)^\alpha} \Phi_s(t).$$

We shall say that $\Phi(t, \rho) \in \Gamma^p(\alpha)$, p an integer ≥ 0 , $\alpha > 1$, if

$$\left(\frac{\partial}{\partial t} \right)^j \lambda \Phi(t, \rho) = \sum_{s=0}^{\infty} \frac{\rho^s}{(s!)^\alpha} \frac{\partial^j \Phi_s}{\partial t^j}, \quad j \leq p,$$

is a holomorphic function of ρ in some neighborhood of zero.

Let $f : X \rightarrow C$, where X is a strip in R^{l+1} , $0 \leq x_0 \leq T$, and C is the complex plane. We shall say that

$$f \in \Upsilon_2^{n(\alpha)} (\Upsilon_{[2]}^{n(\alpha)}),$$

if

$$|D^{n,\infty} f, S_t, \rho| \in \Gamma^0(\alpha) \quad (\|D^{n,\infty} f, S_t, \rho\| \in \Gamma^0(\alpha)).$$

Consider the Cauchy problem

$$P(x, D)u = f; \quad D_0^j u|_{S_0} = 0, \quad j < m; \quad (1)$$

m is the order of P .

Theorem 1. Suppose that $P(x, D)$ has coefficients of the Gevrey class $\gamma_{[2]}^{n(\alpha)}$ and, moreover,

$$P_m(x, \xi) = 0 \quad (*)$$

has, with respect to ξ_0 , the roots

$$\xi_0 = \lambda_1(x, \xi'), \dots, \xi_0 = \lambda_m(x, \xi'); \quad \xi' = (\xi_1, \dots, \xi_l),$$

where $P_m(x, D)$ is the principal part of the operator P .

We shall consider the case of a hyperbolic operator, i.e., all $\lambda_i(x, \xi')$ are real. If the $\lambda_i(x, \xi')$ satisfy the conditions:

1. $\lambda_i(x, \xi') \in C^\infty$ jointly in the variables (x, ξ') for $|\xi'| \neq 0$.
2. There exists

$$\lim_{\substack{x \rightarrow \infty \\ |\xi'| \neq 0}} \lambda_i(x, \xi') = \lambda_i(\infty, \xi')$$

$$\text{and } \lambda_i(x, \xi') - \lambda_i(\infty, \xi') \in S,$$

where S is the Shilov space.

3. $\lambda_i(x, \xi'/|\xi'|) \in \gamma_{[2]}^{n(\alpha)}(X)$ for all $|\xi'| \neq 0$, as functions of x .

Then the Cauchy problem (1) has a unique solution $u \in \gamma_2^{n(\alpha)}(Y)$ for every function $f \in \gamma_2^{n(\alpha)}(X)$; Y is the strip $0 \leq x_0 \leq T' \leq T$, for $1 \leq \alpha \leq m/(m-1)$.

The **proof** differs from the proof of the main theorem in (1) only in that the operator P is replaced by

$$P(x, D) \equiv \Lambda_1 \dots \Lambda_m + \Lambda,$$

where Λ_i are pseudodifferential operators with symbol

$$\xi_0 - \lambda_i(x, \xi'), \quad (2)$$

and Λ is a pseudodifferential operator of order $m-1$.

Preliminarily, for the operator with symbol (2), Gårding's estimate (2) is proved.

Theorem 2. Under the assumptions of Theorem 1, for $1 < \alpha \leq m/(m-1)$ there is a finite domain of dependence.

Remark 1. In the case of constant coefficients, Theorem 1 is known and is due to Gårding.

Theorem 3. Under the assumptions of Theorem 1, there exists a unique solution of the Cauchy problem (1),

$$u \in \gamma_2^{m+n-q(\alpha)}(Y),$$

for every function $f \in \gamma_2^{n(\alpha)}(X)$, in the strip Y in R^{l+1} :

$$0 \leq x_0 \leq T' \leq T;$$

q is the number of non-simple roots of equation (*).

The **proof** is based on representing the operator P in the form

$$P = AB + C, \quad (3)$$

where A is strictly hyperbolic, B is a hyperbolic pseudodifferential operator, and C has arbitrarily small order. The decomposition is constructed as follows: from the roots $\xi_0 - \lambda_1(x, \xi'), \dots, \xi_0 - \lambda_m(x, \xi')$, choose $\xi_0 - \lambda_1, \dots, \xi_0 - \lambda_r$ such that

$$\lambda_i(x, \xi') \neq \lambda_j(x, \xi'), \quad i = 1, \dots, r; \quad j = 1, \dots, m, \quad |\xi'| \neq 0.$$

Form the functions

$$\nu(x, \xi) = \prod_{i=1}^r [\xi_0 - \lambda_i(x, \xi')]$$

and

$$\mu(x, \xi) = \prod_{i>r} [\xi_0 - \lambda_i(x, \xi')].$$

Obviously, $\nu^2 + \mu^2 \neq 0$ for $|\xi| \neq 0$. Assuming that the symbol of A is

$$a(x, \xi) = a_r(x, \xi) + \dots$$

and the symbol of B is

$$b(x, \xi) = b_{m-r}(x, \xi) + b_{m-r-1}(x, \xi) + \dots,$$

we set

$$a_r = \nu(x, \xi), \quad b_{m-r} = \mu(x, \xi).$$

Using the formula for the symbol of the superposition of two operators (4), we see that a_{r-1} and b_{m-r-1} can be chosen as follows:

$$a_{r-1}(x, \xi) = \mu(x, \xi)Q(x, \xi)/\nu^r(x, \xi) + \mu^2(x, \xi),$$

$$b_{m-r-1}(x, \xi) = \nu(x, \xi)Q(x, \xi)/\mu^r(x, \xi) + \nu^2(x, \xi),$$

where

$$Q = P_{m-1}(x, \xi) + \sum_{i=0}^l \frac{\partial \nu(x, \xi)}{\partial x_i} \frac{\partial \mu(x, \xi)}{\partial \xi_i},$$

and, with this choice of $a_{r-1}(x, \xi)$, $b_{m-r-1}(x, \xi)$, the order of the operator $(P - AB)$ is at most $m - 2$. Analogously one can choose also a_{r-2} , b_{m-r-2} , etc. It remains to apply induction on the order m of the equation under the assumption $q < m$ and to use Gårding's inequality for pseudodifferential operators.

Theorem 4. Under the hypotheses of Theorem 1, there exists a unique solution of the Cauchy problem (1) $u \in \gamma_2^{n(\alpha)}$ for any function $f \in \gamma_r^{n(\alpha)}$ when $1 < \alpha \leq q/(q-1)$, where q is the maximal multiplicity of a root of the equation $P_m(x, \xi) = 0$.

Proof is based on a decomposition similar to decomposition (3):

$$P \equiv AB_1 \dots B_s + C,$$

where A is strictly hyperbolic, B_i are hyperbolic pseudodifferential operators, and the order of C is arbitrarily small.

Let us separate among the roots $\xi_0 - \lambda_{r+1}, \dots, \xi_0 - \lambda_m$ the series:

$$\xi_0 - \lambda_1^{(1)}, \dots, \xi_0 - \lambda_{r_1}^{(1)}; \quad \xi_0 - \lambda_1^{(r)}, \dots, \xi_0 - \lambda_{r_2}^{(r)}; \dots$$

such that

$$\lambda_i^{(s)}(x, \xi') \neq \lambda_j^{(k)}(x, \xi')$$

for $s \neq k$, $i = 1, \dots, r_s$, $j = 1, \dots, r_k$, $x \in V$, $\xi' \in R^l$, $|\xi'| \neq 0$;
 V is some neighborhood of the point x_0 .

The operators B_i are constructed from these roots. After this, for the proof it remains to use a partition of unity in the space $\xi' = (\xi_1, \dots, \xi_l)$, induction on m under the assumption $q < m$, and so on.

Remark 2. The sharpness of the result of Theorem 4 for the whole class of hyperbolic equations whose characteristic polynomials have roots of multiplicity q can be proved analogously to the corresponding example in (3).

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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