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Abstract

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MATHEMATICS

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THE FUNDAMENTAL MATRIX OF AN ELLIPTIC SYSTEM OF SECOND ORDER WITH A COMPLEX PARAMETER

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The fundamental matrix for a general elliptic system without a parameter was constructed in the work ⁽¹⁾, whose method was also applied in the case of an elliptic system with a real parameter ⁽²⁾.

In connection with the application of the contour-integral method ⁽³⁾ to the solution of parabolic problems, the present note is devoted to the construction and estimation of the fundamental matrix for the system

$$A(x)\Delta u + \sum_{i=1}^3 A_i(x) \frac{\partial u}{\partial x_i} + (A_0(x) - \lambda^2)u = 0, \quad (1)$$

considered in a three-dimensional domain D of Euclidean space, where $A(x), A_i(x)$ ($i = 0, 1, 2, 3$) are square matrices of order m .

It is assumed that in the closed domain D the matrices $A(x), A_i(x)$ ($i = 0, 1, 2, 3$) have continuous first-order derivatives with respect to all their arguments and that the roots ν_i ($i = 1, 2, \dots, p$) of the characteristic equation

$$\Delta(1, \nu) = \det(A(x) + \nu E) = 0 \quad (2)$$

have constant multiplicity m_i and strictly negative real parts*

$$\operatorname{Re} \nu_i(x) < 0, \quad (3)$$

where $\Delta(\beta, \nu) = \det(\beta A(x) + \nu E)$, and E is the identity matrix of order m .

The fundamental matrix $\tilde{P}_0(x - \xi, \xi, \lambda)$, with singularity at the point $x = \xi$, for the system

$$\sum_{i=1}^3 A(\xi) \frac{\partial^2 u}{\partial x_i^2} - \lambda^2 u = 0$$

is constructed in finite form

$$P_0(x - \xi, \xi, \lambda) = (P_{0ks}(x - \xi, \xi, \lambda))_{k,s=1}^m, \quad (4)$$

where the elements $P_{0ks}(x - \xi, \xi, \lambda)$ admit the representations

$$P_{0ks}(x - \xi, \xi, \lambda) = \frac{1}{4\pi|x - \xi|} \sum_{i=1}^p \sum_{j=1}^{m_i} \frac{B_{sk}^{(i,j)}(\xi)(m_i - j)!}{(-\nu_i(\xi))^{m_i - j + 1}} \times \left\{ \frac{1}{(m_i - j)!} \exp \left[-\lambda \frac{|x - \xi|}{\sqrt{-\nu_i(\xi)}} \right] + \sum_{r=1}^{m_i - j} \frac{1}{(r!)^2 (m_i - j - r)!} \left[\left(-\frac{|x - \xi|}{2\sqrt{-\nu_i(\xi)}} \right)^r \lambda^r + \right. \right.$$

* The fundamental matrices for such an equation in the case of a constant matrix A of second and third orders were constructed in ^(4,5).

$$+ \sum_{q=1}^r (-1)^q \left(\frac{|x - \xi|}{2\sqrt{-v_i(\xi)}} \right)^{r-q} \sum_{j_q=1}^{r-1} \sum_{j_{q-1}=q-1}^{j_q-1} \dots \sum_{j_1=1}^{j_2-1} \left(\frac{j_{\nu+1} + \nu}{2} \right) \lambda^{r-q} \Big] \times \exp \left(-\lambda \frac{|x - \xi|}{\sqrt{-v_i(\xi)}} \right); \quad (5)$$

$|x - \xi|$ is the length of the vector $x - \xi$,

$$B_{sk}^{(i,j)}(\xi) = \frac{1}{(j-1)!} \frac{\partial^{j-1}}{\partial v^{j-1}} \frac{\Delta_{sk}(1, v)}{\prod_{\substack{r=1 \\ r \neq i}}^p (v - v_r(\xi))} \Bigg|_{v=v_i(\xi)} \quad (j = 1, \dots, m_i), \quad (6)$$

$\Delta_{ks}(\beta, \gamma)$ is the cofactor of the element (k, s) in the determinant $\Delta(\beta, \gamma)$.

Formula (5) can be written in a more transparent form:

$$P_{0ks}(x - \xi, \xi, \lambda) =$$

$$= -\frac{1}{4\pi|x-\xi|} \sum_{i=1}^p \sum_{j=1}^{m_i} \frac{B_{sk}^{(i,j)}(\xi)}{(m_i-j+1)!} \frac{\partial^{m_i-j}}{\partial v^{m_i-j}} \frac{\exp\left[-\lambda \frac{|x-\xi|}{\sqrt{-v}}\right]}{v} \Bigg|_{v=v_i(\xi)}. \quad (7)$$

In particular, if all roots $v_i(x)$ of the characteristic equation (2) are simple ($m_i = 1$; $i = 1, 2, \dots, m = p$), then from (7), taking (6) into account, we have

$$P_{0ks}(x-\xi, \xi, \lambda) = -\frac{1}{4\pi|x-\xi|} \sum_{i=1}^m \frac{\Delta_{sk}(1, v_i(\xi))}{\prod_{\substack{r=1 \\ r \neq i}}^p (v_i(\xi) - v_r(\xi))} \exp\left[-\lambda \frac{|x-\xi|}{\sqrt{-v_i(\xi)}}\right]. \quad (8)$$

According to the condition on the roots $v_i(\xi)$ of the characteristic equation, the inequalities (3) hold, and, consequently, there exists a positive number δ such that for all $x, \xi \in \overline{D}$ the estimates

$$\left| \frac{\partial^k P_0(x-\xi, \xi, \lambda)}{\partial x_i^k} \right| \leq \frac{CB \exp(-\varepsilon|\lambda||x-\xi|)}{|x-\xi|^{k+1}} \quad (k = 0, 1, 2); \quad (9)$$

$$\left| \frac{\partial P_0(x-\xi, \xi, \lambda)}{\partial \xi_i} \right| \leq \frac{CB \exp(-2\varepsilon|\lambda||x-\xi|)}{|x-\xi|^2}; \quad (10)$$

$$\frac{\partial^k}{\partial x_i^k} \left(\frac{\partial P_0(x-\xi, \xi, \lambda)}{\partial x_i} + \frac{\partial P_0(x-\xi, \xi, \lambda)}{\partial \xi_i} \right) \leq \frac{CB \exp(-\varepsilon|\lambda||x-\xi|)}{|x-\xi|^{1+k}} \quad (k = 0, 1), \quad (11)$$

where λ is any value from the domain R_δ :

$$|\lambda| \geq R, \quad |\arg \lambda| \leq \pi/4 + \delta; \quad (R_\delta)$$

C, R are sufficiently large positive constants; B is a square matrix of order m , composed of ones; ε is some positive constant; inequalities (9)–(11) hold between the corresponding elements of the left- and right-hand sides.

The fundamental matrix $P(x, \xi, \lambda)$ of system (1), with a singularity at the point $x = \xi$, is constructed by the Levi–Carleman method ^(6,7) in the form

$$P(x, \xi, \lambda) = P_0(x-\xi, \xi, \lambda) + \int_D P_0(x-\eta, \eta, \lambda) h(\eta, \xi, \lambda) dD_\eta, \quad (12)$$

where $h(\eta, \xi, \lambda)$ is the unknown density of the integral correction

$$P_1(x, \xi, \lambda) = \int_D P_0(x - \eta, \eta, \lambda) h(\eta, \xi, \lambda) dD_\eta. \quad (13)$$

Substituting (12) into the left-hand side of equation (1) and (taking into account (9)–(11)) equating the resulting expression to zero, we arrive at the integral equation

$$h(x, \xi, \lambda) = K(x, \xi, \lambda) + \int_D K(x, \eta, \lambda) h(\eta, \xi, \lambda) dD_\eta, \quad (14)$$

where

$$K(x, \xi, \lambda) = \left\{ (A(x) - A(\xi)) \Delta_x + \sum_{i=1}^3 A_i(x) \frac{\partial}{\partial x_i} + A_0(x) \right\} P_0(x - \xi, \xi, \lambda).$$

According to (9), under the restrictions imposed, the following inequality holds for the kernel:

$$|K(x, \xi, \lambda)| \leq CB \exp(-2\varepsilon|\lambda||x - \xi|/|x - \xi|^2), \quad (15)$$

which is satisfied in the domain R_δ .

Estimate (15) makes it possible to construct a solution $h(x, \xi, \lambda)$ of the integral equation (14) in the domain R_δ by the method of successive approximations,

$$h(x, \xi, \lambda) = K(x, \xi, \lambda) + \sum_{n=2}^{\infty} K_n(x, \xi, \lambda), \quad (16)$$

where K_n are the iterations of the kernel K ; moreover, for h the estimate

$$|h(x, \xi, \lambda)| \leq CB \exp(-\varepsilon|\lambda||x - \xi|/|x - \xi|^2) \quad (17)$$

holds in the domain R_δ .

Thus one proves the

Theorem. *Under the restrictions imposed, there exists a positive number δ such that, for sufficiently large R , for all complex values of λ satisfying the inequalities (R_δ) , the system [1] has a fundamental matrix $P(x, \xi, \lambda)$ of the form (12), analytic in λ in the domain R_δ , where for $x, \xi \in \overline{D}$ the inequalities (9), (11) and the estimates*

$$|P_1(x, \xi, \lambda)| \leq CB \exp(-\varepsilon|\lambda||x - \xi|/|\lambda|x - \xi|,$$

$$|\partial^k P_1(x, \xi, \lambda) / \partial x_i^k| \leq CB \exp(-\varepsilon |\lambda| |x - \xi|) / |x - \xi|^k \quad (k = 1, 2),$$

$$\left| \frac{\partial^k}{\partial x_i^k} \left[\frac{\partial P(x, \xi, \lambda)}{\partial x_i} + \frac{\partial P_0(x - \xi, \xi, \lambda)}{\partial \xi_i} \right] \right| \leq \frac{CB \exp(-\varepsilon |\lambda| |x - \xi|)}{|x - \xi|^{1+k}},$$

$$|\partial^k P(x, \xi, \lambda) / \partial x_i^k| \leq CB \exp(-\varepsilon |\lambda| |x - \xi|) / |x - \xi|^{1+k} \quad (k = 0, 1, 2).$$

If $\Phi(x)$ is a vector-function having in D bounded continuous first-order derivatives with respect to all its arguments, then the vector-function

$$u(x, \lambda, \Phi) = - \int_D P(x, \xi, \lambda) \Phi(\xi) dD_\xi$$

for all λ belonging to R_δ is a solution of the nonhomogeneous equation

$$A(x)\Delta u + \sum_{i=1}^3 A_i(x) \frac{\partial u}{\partial x_i} + (A_0(x) - \lambda^2)u = \Phi(x).$$

An analogous theorem also holds for the case of a domain D of an arbitrary number of dimensions greater than one, with the corresponding complications of the formulas and estimates given, and $P_{0ks}(x - \xi, \xi, \lambda)$ is expressed in terms of a Bessel function.

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