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Abstract

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MATHEMATICS

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ON THE STABILITY OF SELF-OSCILLATIONS OF A FLUID

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A linearization law is established in the stability problem for a periodic self-oscillation of a viscous incompressible fluid. An infinite-dimensional analogue of the Andronov–Witt theorem ^(1,2) is proved.

1. The general stability problem for a cycle. Let a dynamical system (Q_t, X) be given, i.e., a nonlinear partial semigroup of operators $Q_t : X \rightarrow X$ ($0 \leq t < \infty$), acting in a Banach space X . This means that for each $x \in X$ an element $Q_t x$ is defined for $0 \leq t < t_0(x)$, with $Q_0 x = x$; $Q_{t+\tau} x = Q_t Q_\tau x$ ($t, \tau \geq 0$; $t + \tau < t_0(x)$).

Suppose that the operator Q_T for some $T > 0$ has a fixed point q_0 . Then $q_\tau = Q_\tau q_0$ is also a fixed point of the operator Q_T for all $\tau > 0$. If $C = \bigcup_{0 \leq \tau \leq T} q_\tau$ contains more than one point, we shall say that C is a cycle (or self-oscillation) of the dynamical system (Q_t, X) . We shall call it (asymptotically) stable if, for every $\varepsilon > 0$, one can indicate such a $\delta > 0$ that from $\rho(x_0, C) < \delta$ it follows that $t_0(x_0) = \infty$ and $\rho(Q_t x_0, C) < \varepsilon$ for $t \geq 0$ (and $\rho(Q_t x_0, C) \rightarrow 0$ as $t \rightarrow +\infty$). If $\|Q_t x_0 - q_{t+h}\| \rightarrow 0$ as $t \rightarrow +\infty$, then we shall say that the trajectory $Q_t x_0$ has asymptotic phase h .

Next, we shall call the cycle C smooth if in some neighborhood of it $\omega_\eta = \{x \in X : \rho(x, C) < \eta\}$ the operators Q_t are continuously Fréchet differentiable (with respect to x), the derivative $d/d\tau q_\tau = \dot{q}_\tau$ exists ($0 \leq \tau \leq T$), and, moreover, for any $\varepsilon_1, \varepsilon_2 > 0$ there exist $\delta_1, \delta_2 > 0$ such that from $\|a\| < \delta_1$, $|s| < \delta_2$ follow estimates, uniform in $\tau \in [0, T]$,

$$\|\Delta_1(\tau, a)\| \equiv \|Q_T(q_\tau + a) - q_\tau - Q'_T(q_\tau)a\| \leq \varepsilon_1 \|a\|; \quad (1)$$

$$\|\Delta_2(\tau, s)\| \equiv \|q_{\tau+s} - q_\tau - \dot{q}_\tau s\| \leq \varepsilon_2 |s|. \quad (2)$$

The operator $U_{T,\tau} = Q'_T(q_\tau)$ will be called the monodromy operator (corresponding to the initial instant τ). Differentiating the equality $Q_T q_\tau = q_\tau$ with respect

to τ , we infer that $\varphi_\tau = \dot{q}_\tau$ is an eigenvector of the operator $U_{T,\tau}$ corresponding to the eigenvalue 1: $U_{T,\tau}\varphi_\tau = \varphi_\tau$.

Lemma 1. *The spectrum of the monodromy operator $U_{T,\tau}$ does not depend on τ .*

This lemma is easily derived by taking into account the equalities

$$U_{T,\tau} = Q'_\tau(q_0)Q'_{T-\tau}(q_\tau), \quad U_{T,0} = Q'_{T-\tau}(q_\tau)Q'_\tau(q_0) \quad (3)$$

and using the following simple assertion.

Lemma 2. *Let $U, V : X \rightarrow X$ be linear bounded operators. Then the operators UV and VU have one and the same nonzero spectrum: $\sigma(UV) - \{0\} = \sigma(VU) - \{0\}$. Moreover, if 0 is excluded, the point, continuous, and residual spectra, respectively, coincide. The multiplicities of the nonzero eigenvalues are then the same.*

Theorem 1. *Let $C = \bigcup_{0 \leq \tau \leq T} Q_\tau q_0$ be a smooth cycle of the dynamical system (Q_t, X) , and let the spectrum of the monodromy operator $U_{T,\tau}$ have the form*

$$\sigma(U_{T,\tau}) = \{1\} \cup \sigma_0(U_{T,\tau}), \quad |\sigma_0(U_{T,\tau})| < \alpha < 1. \quad (4)$$

Let 1 be a simple proper number. Then the cycle C is asymptotically stable, and every trajectory $\{Q_t x^0\}$, $t \geq 0$, has an asymptotic phase, provided only that the quantity $\rho(x_0, C)$ is sufficiently small.

Proof. Let ψ_τ be a fixed vector of the adjoint operator $U_{T,\tau}^*$, normalized by the condition $(\varphi_\tau, \psi_\tau) = 1$. Define the operator V_τ by setting

$$V_\tau x = U_{T,\tau} x - (x, \psi_\tau) \varphi_\tau. \quad (5)$$

It is clear that $\sigma(V_\tau) = \sigma_0(U_{T,\tau})$. Therefore, for sufficiently large natural m , the operator V_τ^m is a contraction (uniformly in τ): $\|V_\tau^m\| < \theta < 1$. We shall assume that this already holds for $m = 1$; this case can be attained by introducing the substitution $T \rightarrow mT$.

Let $\rho(x_0, C) < \delta$. Then for some τ_0 , $0 \leq \tau_0 \leq T$, we have

$$\|x_0 - q_{\tau_0}\| < \delta. \quad (6)$$

Define sequences of time instants τ_n and elements a_n of the space X by setting

$$\tau_{n+1} = \tau_n + s_n, \quad s_n = (a_n, \psi_{\tau_n}), \quad a_n = x_n - q_{\tau_n}, \quad n = 0, 1, \dots \quad (7)$$

We shall show that, if δ is sufficiently small, then the estimates

$$\rho(Q_T^n x_0, C) \leq \|a_n\| \leq \theta^n \delta, \quad |s_n| \leq l\theta^n \delta, \quad l = \max \|\psi_\tau\| \quad (8)$$

hold.

For $n = 0$, the estimates (8) follow immediately from (6). If the estimates (8) have already been proved for $n = k$, then for $n = k + 1$ we derive them, using conditions (1) and (2), from the relation

$$\rho(x_{k+1}, C) \leq \|a_{k+1}\| = \|V_{\tau_k} a_k + \Delta_1(\tau_k, a_k) + \Delta_2(\tau_k, s_k)\|.$$

Here it is sufficient to choose $\varepsilon_1, \varepsilon_2, \delta$ so small that the inequalities

$$\max_{\tau} \|V_{\tau}\| + \varepsilon_1 + l\varepsilon_2 < \theta, \quad \delta < \delta_1, \quad l\delta < \delta_2 \quad (9)$$

are satisfied.

It is now not difficult to establish that

$$\|Q_t x_0 - q_{h+t}\| \rightarrow 0 \quad (t \rightarrow +\infty), \quad h = \tau_0 + \sum_{n=0}^{\infty} s_n, \quad (10)$$

which completes the proof.

Theorem 2. *Let C be a smooth cycle of the dynamical system (Q_t, X) . Let the mapping Q_t be differentiable with respect to t , and let the derivative $\frac{d}{dt}Q_t = \dot{Q}_t$ be continuous in (x, t) : $x \in \omega_n$, $0 \leq t < t_0(x)$. Let the spectrum of the monodromy operator have the form*

$$\sigma(U_{T,\tau}) = \sigma_1(U_{T,\tau}) \cup \sigma_2(U_{T,\tau}); \quad |\sigma_1(U_{T,\tau})| > \beta > 1, \quad |\sigma_2(U_{T,\tau})| \leq 1. \quad (11)$$

Then the cycle C is unstable.

Proof. Let $\psi \in X^*$ and $(\varphi_0, \psi) = 1$ (it is easy to prove that $\varphi_0 = \dot{q}_0 \neq 0$). Consider in the space X the plane $X_0 = \{a \in X : (a, \psi) = 0\}$ and the hyperplane $\Gamma = \{x : x = q_0 + a; a \in X_0\}$. Define a mapping K of a neighborhood of zero in the space X_0 into X_0 by setting

$$Ka = Q_{t_*}(q_0 + a) - q_0. \quad (12)$$

Here $t_* = t_*(x)$ is the instant of the first return of the trajectory $Q_t x$, $x \in \Gamma$, to the hyperplane Γ . To prove the existence of t_* , it is sufficient to apply the implicit-function theorem to the equation

$$F(t, a) \equiv (Q_t(q_0 + a) - q_0, \psi) = 0. \quad (13)$$

Indeed, $F(T, 0) = 0$, $F_t(T, 0) = (\dot{q}_0, \psi) = 1$, and the function F is continuously differentiable in a neighborhood of the point $(T, 0) \in R \times X_0$.

The operator K is continuously differentiable in a neighborhood of zero, and

$$K'(0)a = U_{\tau,0}a - \varphi_0(U_{\tau,0}a, \psi), \quad a \in X_0. \quad (14)$$

The spectrum of the operator $K'(0)$ obviously contains the set $\sigma_1(U_{\tau,\tau})$. Now Theorem 2 is easily derived from Lemma 5 of paper ⁽³⁾, if one further notes that for points $x = q_0 + a$, $a \in X_0$; $\|a\| < \varepsilon$, for sufficiently small $\varepsilon > 0$, there exists a constant $\gamma > 0$ such that $\rho(x, C) \geq \gamma\|a\|$.

2. Application to the Navier–Stokes equations. Let a viscous incompressible homogeneous fluid fill a three-dimensional bounded domain Ω with boundary S of class C^2 . Let the body forces and the boundary value of the velocity be prescribed and independent of time. Then the Navier–Stokes equations and boundary conditions have the form

$$v_t + (v, \nabla)v - \nu \Delta v = -\nabla P + F(x), \quad x \in \Omega; \quad (15)$$

$$\operatorname{div} v = 0; \quad (16)$$

$$v|_S = a(x). \quad (17)$$

Suppose that there exists a (sufficiently smooth) T -periodic in time t self-oscillatory solution of system (15)–(17), with velocity vector $v_0(x, t)$ and pressure $P_0(x, t)$. We shall be interested in its stability with respect to perturbations from the Hilbert space H_1 . The latter is the closure of the set of smooth solenoidal vector fields vanishing on the boundary in the metric

$$(u, v)_{H_1} = \int_{\Omega} \sum_{k=1}^3 \frac{\partial u}{\partial x_k} \frac{\partial v}{\partial x_k} dx. \quad (18)$$

We shall call the cycle $C = \bigcup_t v_0(\cdot, t)$ (asymptotically) stable in H_1 , if for every $\varepsilon > 0$ there corresponds a $\delta > 0$ such that

$$\rho(v(\cdot, t), C) = \inf_{\tau} \|v(\cdot, t) - v_0(\cdot, \tau)\|_{H_1} < \varepsilon$$

for any solution of system (15)–(17) with velocity vector v , provided that $\rho(v(\cdot, 0), C) < \delta$.

The stability spectrum $\Sigma(v_0)$ ⁽³⁾ is the set of those σ for which there exists a nonzero T -periodic solution of the linearized system

$$u_t + \sigma u + (v_0, \nabla)u + (u, \nabla)v_0 - \nu \Delta u = -\nabla q, \quad \operatorname{div} u = 0, \quad u|_S = 0. \quad (19)$$

Obviously, $\sigma_k = -2k\pi i/T \in \Sigma(v_0)$ ($k = 0, \pm 1, \dots$): the corresponding solution of system (19) is $v_{0t} \exp \sigma_k t$, $P_{0t} \exp \sigma_k t$. Applying Theorems 1 and 2, we arrive at the following conclusions.

Theorem 3. *Let $\sigma_0 = 0$ be a simple eigenvalue of system (19), and let all points of the stability spectrum $\Sigma(v_0)$, except $\sigma_k : k = 0, \pm 1, \dots$, lie inside the left half-plane. Then the cycle C is asymptotically stable in H_1 .*

Theorem 4. *If the stability spectrum $\Sigma(v_0)$ contains at least one point σ with $\operatorname{Re} \sigma > 0$, then the cycle C is unstable.*

For examples of self-oscillatory regimes, see ^(4–7).

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Note: Figure translations are in progress. See original paper for figures.

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