

JUSTIFICATION OF THE AVERAGING METHOD

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Abstract

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MATHEMATICS

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JUSTIFICATION OF THE AVERAGING METHOD

FOR ABSTRACT PARABOLIC EQUATIONS

(Presented by Academician N. N. Bogolyubov on 18 VII 1969)

In the present paper, N. N. Bogolyubov's averaging method¹ is applied to abstract parabolic equations of the form

$$dx/dt = Ax + f(x, \omega t), \quad (1)$$

where A is a linear, generally speaking unbounded, operator generating an analytic semigroup.

It is assumed that: 1) $-A$ is a positive operator possessing a completely continuous inverse*.

- 2) f is a continuous mapping of the space B^α ($\alpha \in [0, 1)$) into the space B^ε ($\varepsilon > 0$). By B^α we denote the Banach space that is the domain of definition of the operator $(-A)^\alpha$.
- 3) The Fréchet differential with respect to the first variable exists and defines a continuous mapping Df of the space $B^\alpha \times [0, +\infty)$ into the space $\text{Hom}(B^\alpha, B^\varepsilon)$. (Thus we denote the space of bounded linear operators acting from the space B^α into the space B^ε with the topology of uniform convergence.) For every bounded set K in B^α , the mapping Df is uniformly continuous and bounded on the subset $K \times [0, +\infty)$.
- 4) The set where the integral

$$(D_N f)(x) = \frac{1}{N} \int_0^N Df(x, t) dt \quad (N \rightarrow +\infty),$$

converges in $\text{Hom}(B^\alpha, B^\varepsilon)$, is everywhere dense in B^α .

- 5) There exists an element $x \in B^\alpha$ such that the integral

$$F_N x = \frac{1}{N} \int_0^N f(x, t) dt \quad (N \rightarrow \infty),$$

converges in B^ε , and there exists a constant M such that $\|f(x, t)\|_{B^\varepsilon} \leq M$ for all $t \geq 0$. We denote the limit of $F_N x$ by Fx .

1°. Consider the two Cauchy problems

$$dx/dt = Ax + f(x, \omega t), \quad x|_{t=0} = x_0, \quad \omega > 0; \quad (2)$$

$$dy/dt = Ay + Fy, \quad y|_{t=0} = x_0. \quad (3)$$

With respect to the initial data we shall suppose that $x_0 \in B^\beta$, $\alpha < \beta < 1$. We shall seek the solution in the class $C_{[0, t_0]}(B^\beta)$. This denotes the space of continuous mappings of the segment $[0, t_0]$ into the space B^β . The equations are satisfied for $t \in (0, t_0]$.

In this formulation, the problems (2), (3) can have only a unique solution. Solvability, generally speaking, holds only locally, i.e. for small t_0 .

* We use, without further qualifications, notions from the theory of semigroups and fractional powers of an operator ^{2,3}. Let us also note that the condition of positivity of the operator $-A$ is inessential, since for any operator A generating an analytic semigroup there exists a number $\lambda(> 0)$ such that $\lambda I - A$ is positive, and we may replace A by $\lambda I - A$, including λI in f .

2°. **Theorem 1.** If problem (3) is solvable, then there exists a number ω_0 such that for $\omega > \omega_0$ problem (2) is also solvable. Moreover,

$$\lim_{\omega \rightarrow +\infty} \|x - y\|_{C_{[0, t_0]}(B)} = 0.$$

3°. **Theorem 2.** Let y be a stationary solution of equation (3), and let the mapping f be T -periodic in the second variable ($f(x, t + T) = f(x, t)$).

Denote by Λ the spectrum of the operator $A + DF(\dot{y})$, $r(\Lambda) = \max_{\lambda \in \Lambda} \operatorname{Re} \lambda$. Suppose that the spectrum Λ does not contain zero. Then:

- 1) To the stationary solution $\dot{y}$, at large frequencies, there corresponds a unique $\omega^{-1}T$ -periodic solution of equation* (2), i.e., there exist numbers $\varepsilon(> 0)$, ω_ε , such that, under the condition $\omega > \omega_\varepsilon$, there exists a unique $\omega^{-1}T$ -periodic solution $\dot{x}$ of equation (2) satisfying the inequality

$$\|\dot{y} - \dot{x}\|_{C_{[0, t_0]}(B^\beta)} < \varepsilon.$$

- 2) If the entire spectrum Λ is situated strictly in the left half-plane, then the indicated $\omega^{-1}T$ -periodic solution of equation (2) is asymptotically stable for large ω . This means that for every $\sigma \in (0, -r(\Lambda))$ there exist numbers $\omega_0(> \omega_\varepsilon)$, c , $\varepsilon_1(> 0)$ such that, for $\omega > \omega_0$, for any initial data x_0 from the neighborhood

$$\|\dot{y} - x_0\|_{B^\beta} < \varepsilon_1$$

the solution x of problem (2) exists on an infinite time interval and the estimate holds

$$\rho(t) = \|\dot{x}(t) - x(t)\|_{B^\beta} \leq ce^{-\sigma t} \rho(0).$$

- 3) If the spectrum Λ has at least one point strictly in the right half-plane, then the corresponding periodic solution of equation (2) is unstable for large ω . This means that for every $\sigma \in (0, r(\Lambda))$ there exist numbers $\varepsilon_1(> 0)$, $\omega_0(> \omega_\varepsilon)$, a sequence $t_n(> 0)$, and a sequence of initial data x_0^n , converging in B^β to the initial data of the $\omega^{-1}T$ -periodic solution $\dot{x}$, for which problem (2) is solvable for $\omega > \omega_0$ on the time interval $[0, t_n]$, and the inequalities

$$\rho_n(t_n) \geq \varepsilon_1, \quad t_n \leq \frac{1}{\sigma} \ln \frac{\varepsilon_1}{\rho_n(0)}, \quad \rho_n(t) = \|x_n(t) - \dot{x}(t)\|_{B^\beta}.$$

hold. Here x_n is the solution of problem (2) with initial data x_0^n .

4°. The proof of the theorems is based on the theory of semigroups, fractional powers of an operator ^(2,3), and the theory of stability in Banach spaces ⁽⁴⁾.

5°. The formulated theorems contain new results, for example, for parabolic equations (A an elliptic operator), and for the Navier–Stokes equations (in connection with the generating property of the principal part of the operator ⁽⁵⁾).

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CITED LITERATURE

- ¹ N. N. Bogolyubov, Yu. A. Mitropolskii, *Asymptotic Methods in the Theory of Nonlinear Oscillations*, 1955.
² M. A. Krasnosel'skii, P. P. Zabreiko et al., *Integral Operators in Spaces of Summable Functions*, "Nauka," 1966.

³ S. G. Krein, *Linear Differential Equations in Banach Space*, “Nauka,” 1967.

⁴ M. G. Krein, *Lectures on the Theory of Stability of Differential Equations in Banach Space*, 1964.

⁵ V. I. Yudovich, DAN, 161, No. 5, 1037 (1965).

* When we speak of an equation, we omit the initial conditions.

Note: Figure translations are in progress. See original paper for figures.

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