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1970

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Abstract

Full Text

UDC 517.93

MATHEMATICS

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CONSTRUCTION OF SEPARATRIX SURFACES AND REGIONS OF STABILITY IN THE PHASE SPACES OF NONLINEAR SYSTEMS

1. Consider the system

$$\dot{\eta} = A\eta + bf(\sigma), \quad \sigma = c'\eta, \quad (1)$$

where η is a column of n variables η_k ; $A = (a_{ki})$ is an $n \times n$ matrix; b, c are columns of n constants b_k, c_k (the prime denotes transposition); $f(\sigma)$ is a scalar function; $\dot{} = d/dt$; t is the independent variable (time).

By means of the transformation

$$\eta = Kx \quad (2)$$

we pass from system (1) to the system

$$\dot{x} = \Lambda x + ef(\sigma), \quad \sigma = \gamma'x. \quad (3)$$

Here x is a column of n variables x_i ; $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$; $e = (1, \dots, 1)$ is an n -dimensional column; γ' is a row of n quantities γ_i ;

$$\gamma_i = -[D'(\lambda_i)]^{-1} \sum_{k=1}^n c_k N_k(\lambda_i); \quad N_i(\lambda_k) = \sum_{\alpha=1}^n b_\alpha D_{\alpha i}(\lambda_k), \quad i = 1, \dots, n; \quad (4)$$

λ_i are the roots of the equation $D(\lambda) \equiv \det[a_{ki} - \delta_{ki}\lambda] = 0$; $D'(\lambda) = dD(\lambda)/d\lambda$; $D_{\alpha i}(\lambda_k)$ is the algebraic cofactor of the element of row α , column i , of the determinant $D(\lambda_k)$. The matrix K is nonsingular under the conditions indicated in ⁽¹⁾.

Let a_{ki}, b_k be prescribed numbers, and c_k parameters. In the n -dimensional space C_n of parameters $c_1, \dots, c_n$, define the revealing sections $G_2^{(s,r)}$ -planes of dimension 2—by the equations

$$\sum_{k=1}^n c_k N_k(\lambda_i) = \delta_{is} A_s + \delta_{ir} A_r, \quad i = 1, \dots, n, \quad (5)$$

where δ_{ij} is the Kronecker symbol, A_s, A_r are real (if λ_s, λ_r are real) or complex conjugate (if λ_s, λ_r are complex conjugate), and otherwise arbitrary constants. Under the conditions of the section (5) we have $\gamma_i = 0$ ($i \neq s, r$), and equations (3) for x_s, x_r, σ form an independent subsystem T , from which we find $\sigma(t)$; after this, equations (3) for x_i ($i \neq s, r$) are integrated as linear inhomogeneous equations.

2. Consider the space X of the variables x_k ; since a pair of equations with complex $x_i, x_{i+1} = \bar{x}_i \pm j\bar{x}_{i+1}$ corresponds to a pair of equations with real $\bar{x}_i, \bar{x}_{i+1}$, by the space X we shall mean the Euclidean space of the variables $x_1, \dots, \bar{x}_i, \bar{x}_{i+1}, \dots, x_n$.

Under the conditions of the section $G_2^{(s,r)}$, the phase portrait on the plane with Cartesian coordinates x_s, x_r of the subsystem T may be regarded as the projection of the phase portrait in the n -dimensional space X of the complete system (3) onto the plane x_s, x_r of this space. To a limit cycle Γ in the plane x_s, x_r there corresponds a separatrix surface S_Γ of dimension $n - 1$ in the space X , representing a cylindrical surface parallel to the axes x_i ($i \neq s, r$) and forming, in its intersection with the plane x_s, x_r , a closed-

curve Γ . Analogously, the separatrix surfaces $S_c, S_{o.p.}$, generated by separatrices and rest segments of the plane x_s, x_r , are defined.

Let there be an equation of a limit cycle in the plane x_s, x_r

$$S(x_s, x_r) = 0. \quad (6)$$

Then the equation of the separatrix surface S_Γ in the space of the variables $\eta_1, \dots, \eta_n$ will be

$$S \left[(\det K)^{-1} \sum_{i=1}^n K_{is} \eta_i; (\det K)^{-1} \sum_{i=1}^n K_{ir} \eta_i \right] = 0, \quad (7)$$

where K_{ij} is the algebraic cofactor of the element in row i , column j of the determinant of the matrix K .

3. Let an unstable limit cycle Γ bound the region of asymptotic stability of the equilibrium state of the subsystem T in the plane x_s, x_r , and let $\operatorname{Re} \lambda_i < 0$ ($i \neq s, r$); then surface (7) in the section $G_2^{(s,r)}$ serves as the

exact boundary of the domain of attraction of the equilibrium state in the n -dimensional phase space of system (1).

Construct the Lyapunov function

$$V(x_1, \dots, x_n) = \sum_{\substack{i=1 \\ i \neq s, r}}^n h_i x_i^2 + h_r \xi_r^2 + h_s \xi_s^2, \quad (8)$$

where $h_r \xi_r^2 + h_s \xi_s^2 = \text{const}$ is the equation of an ellipse approximating the cycle Γ and determined by the method of harmonic linearization; ξ_s, ξ_r are linear functions of x_s, x_r , determined by reducing the equation of the ellipse to its principal axes; h_i are positive constants.

Using Sylvester's inequalities, we determine the values of the parameters c_k in $G_2^{(s,r)}$ for which $\dot{V} < 0$ in a neighborhood of the origin.

Consider the surfaces

$$F(x_1, \dots, x_n) \equiv \dot{V}(x_1, \dots, x_n) = 0; \quad (9)$$

$$\Phi(x_1, \dots, x_n) \equiv \sum_{\substack{i=1 \\ i \neq s, r}}^n h_i x_i^2 + h_r \xi_r^2 + h_s \xi_s^2 - V_i = 0, \quad (10)$$

where V_i is some number. Define the domain of attraction of the equilibrium state Q_i as the largest domain bounded by surface (10) and wholly inscribed in the domain bounded by surface (9). In particular, for a smooth function $f(\sigma)$, the boundary of the domain Q_i is described by equation (10) for $V_i = V_i^*$, where V_i^* is the smallest of the numbers V_i determined by solving the system

$$\left(\frac{\partial \Phi}{\partial x_i} \right)_* \left(\frac{\partial F}{\partial x_i} \right)_*^{-1} = \text{const}, \quad i = 1, \dots, n; \quad (11)$$

$$F(x_1^*, \dots, x_n^*) = \Phi(x_1^*, \dots, x_n^*) = 0.$$

Under the conditions of the section $G_2^{(s,r)}$, system (11) has a solution as $h_i \rightarrow 0$ ($i \neq s, r$) and determines the domain of attraction of the equilibrium state as the interior of an n -dimensional cylinder.

4. Consider a neighborhood of the section $G_2^{(s,r)}$, where $\gamma_i = \varepsilon_i \leq \varepsilon$ ($i \neq s, r$), $\varepsilon > 0$ is a small number. By an orthogonal transformation of variables

$$x = Lz \quad (12)$$

we reduce system (3) to the form

$$\dot{z} = Dz + gf(\sigma), \quad \sigma = \beta'z. \quad (13)$$

Here $L = (l_{ij})$, $D = (d_{ij})$ are $n \times n$ matrices; g is an n -dimensional column; $\beta' = (0, \dots, \beta_s, \dots, \beta_r, \dots, 0)$ is an n -dimensional row; $\beta_s \beta_r^{-1} = \gamma_s \gamma_r^{-1}$. Elements of the matrices L' satisfy the relations

$$\sum_{j=1}^n l_{jk} \gamma_j = 0, \quad k = 1, \dots, s-1, s+1, \dots, r-1, r+1, \dots, n;$$

$$\sum_{j=1}^n l_{ij} l_{jk} = \delta_{jk}, \quad j, k = 1, \dots, n, \quad (14)$$

and are determined up to $\frac{1}{2}(n^2 - 3n + 2)$ arbitrary constants. These constants can be chosen so that the numbers $|d_{ij}|$ ($i \neq j$) are sufficiently small, and from the conditions $\varepsilon_i \rightarrow 0$ ($i \neq s, r$) it follows that $d_{ij} \rightarrow 0$ ($i \neq j$). By the method of harmonic linearization we determine a cycle Γ in the space Z of the variables $z_1, \dots, z_n$ and consider the ellipse generated by it in the Cartesian coordinates z_s, z_r . After this we determine the function V by formula (8), in which we replace x_i by z_i , and repeat the exposition of item 3. The solution of system (11) will now determine the surface bounding a certain elongated body. The choice of h_i ($i \neq s, r$) makes it possible to construct a family of regions Q_i ; the desired region will be their union. Transformations (12) and (2) make it possible to recalculate this region into the space of the variables $\eta_1, \dots, \eta_n$ of the original system (1).

The method considered for constructing regions of stability finds application in the design of control systems for moving objects ⁽²⁾.

Received 18 XII 1969

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2. B. N. Petrov, B. M. Shamrikov, in the book: *Exact Methods for the Study of Nonlinear Automatic Control Systems*, 1970 (in press).

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