

CLASSIFICATION OF LINEAR REPRESENTATIONS OF THE GROUP

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Abstract

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MATHEMATICS

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CLASSIFICATION OF LINEAR REPRESENTATIONS OF THE GROUP

$SL(2, \mathbb{C})$

0. Irreducible representations of the group $SL(2, \mathbb{C})$ are well known. However, in contrast to compact groups, the study of any of its representations is not reduced to irreducible ones. In the present paper we classify all representations of the group $SL(2, \mathbb{C})$ “composed,” in a natural sense, of a finite number of irreducible ones; we call these representations Harish-Chandra modules (for precise definitions see below). A complete description of such representations is given. Namely, each such representation decomposes into a direct sum of a finite number of further indecomposable ones. In the paper all indecomposable representations are determined up to equivalence. They are of special and nonspecial types, and the special case is the most interesting one (for the formulation of the result see §5). A method is also given for composing indecomposable representations with prescribed invariants from certain elementary blocks, which are simply constructed from irreducible representations. The nonspecial case was previously analyzed by D. P. Zhelobenko ⁽⁶⁾, who also has some further considerations. The present paper is based on works ⁽¹⁻³⁾, in which an infinitesimal classification of Harish-Chandra modules was obtained.

1°. Definition of Harish-Chandra G -modules. A linear topological space H , in which a continuous representation of a connected semisimple Lie group G acts, will be called a G -module. Let U be a compact group. By an algebraic U -module we shall mean a linear space H_0 (without topology), in which a representation of the group U acts and for which the following condition is satisfied: the representation in H_0 decomposes into a direct sum of irreducible (finite-dimensional) representations, each of these representations occurring in the decomposition with finite multiplicity. We shall call a G -module H a Harish-Chandra module if H contains as an everywhere dense subspace an algebraic U -module H_0 , where U is a maximal compact subgroup of the group G . Note that H_0 is the minimal linear subspace of H containing all irreducible U -submodules in H ; thus H_0 is uniquely determined in H .

Let $\mathfrak{g}$ be the Lie algebra of the group G . It can be proved that the operators on H corresponding to elements of $\mathfrak{g}$ are defined on all of $H_0 \subset H$ and that H_0 is invariant with respect to these operators. Thus the space H_0 is endowed with the structure of a $\mathfrak{g}$ -module. A Harish-Chandra G -module H will be called **indecomposable** if the corresponding $\mathfrak{g}$ -module H_0 is indecomposable; we shall call H **irreducible** if the $\mathfrak{g}$ -module H_0 is irreducible. Two Harish-Chandra G -modules H' and H'' will be called **equivalent** if the corresponding $\mathfrak{g}$ -modules H'_0 and H''_0 are isomorphic. Note that both the definition of a Harish-Chandra G -module and the definitions of indecomposability, irreducibility, and equivalence do not depend on the choice of the maximal compact subgroup $U \subset G$.

2°. Elementary G -modules. We construct the simplest class of indecomposable Harish-Chandra G -modules, where $G = SL(2, \mathbb{C})$. Let $\pi = (n_1, n_2)$ be an arbitrary pair of complex numbers whose difference is an integer. We shall say that a function $f(z_1, z_2)$, $z_1, z_2 \in \mathbb{C}$, $(z_1, z_2) \neq (0, 0)$, is **homogeneous of degree π** , if

$$f(\lambda z_1, \lambda z_2) = \lambda^{n_1-1} \bar{\lambda}^{n_2-1} f(z_1, z_2)$$

for every $\lambda \neq 0$. Homogeneous functions will also be called called associated homogeneous functions of order zero.

We shall call a function $f(z_1, z_2)$ an associated homogeneous function of degree π of order m ($m = 1, 2, \dots$) if, for every $\lambda \neq 0$, the difference

$$f(\lambda z_1, \lambda z_2) - \lambda^{n_1-1} \bar{\lambda}^{n_2-1} f(z_1, z_2)$$

is an associated homogeneous function of degree π of order $(m-1)$.

Let D_π^m be the space of all infinitely differentiable associated homogeneous functions of degree π of order m . In D_π^m define a representation $T(g)$ of the group $G = SL(2, \mathbb{C})$ by the following formula:

$$(T(g)f)(z_1, z_2) = f(\alpha z_1 + \gamma z_2, \beta z_1 + \delta z_2), \quad g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

We shall call the constructed G -modules D_π^m **elementary***. It can be proved that all of them are indecomposable. Let us formulate assertions about the structure of their composition series.

It is obvious that

$$D_\pi^0 \subset D_\pi^1 \subset \dots \subset D_\pi^m \subset \dots$$

Moreover, $D_\pi^m / D_\pi^{m-1} \simeq D_\pi^0$ for every m . We shall call a pair $\pi = (n_1, n_2)$ a special point if n_1, n_2 are integers different from zero and, moreover, of the same sign. If π is not a special point, then the G -module D_π^0 is irreducible, and therefore the series

$$0 \subset D_\pi^0 \subset D_\pi^1 \subset \dots \subset D_\pi^m$$

is a composition series for D_π^m . If, however, $\pi = (n_1, n_2)$ is a special point, then for every m there exists, and moreover uniquely, a G -module F_π^m (different from D_π^{m-1} and D_π^m) such that

$$D_\pi^{m-1} \subset F_\pi^m \subset D_\pi^{m+1}.$$

Thus, in the special case the composition series for D_π^m has the form

$$0 \subset F_\pi^0 \subset D_\pi^0 \subset F_\pi^1 \subset D_\pi^1 \subset \dots \subset F_\pi^m \subset D_\pi^{m+1}. \quad (1)$$

The factors of the series (1) are the G -modules F_π^0 and $F_{\pi^{-1}}^0$, where $\pi^{-1} = (-n_1, -n_2)$ (namely,

$$F_\pi^k / D_\pi^{k-1} \simeq F_\pi^0, \quad D_\pi^k / F_\pi^k \simeq F_{\pi^{-1}}^0$$

); one of them is finite-dimensional and the other infinite-dimensional (for example, in the case $n_1 > 0$, $n_2 > 0$, F_π^0 is the (finite-dimensional) module of all homogeneous polynomials of degree π , and $F_{\pi^{-1}}^0 \simeq D_{\pi'}^0$, where $\pi' = (n_1, -n_2)$).

By elementary modules in the special case we shall mean both the G -modules D_π^m and the G -modules F_π^m obtained from them by “shortening.” We note that the G -modules $D_{\pi^{-1}}^{m-1}$ and $F_{\pi^{-1}}^m$ can also be obtained by “shortening” D_π^m , namely:

$$F_{\pi^{-1}}^m \simeq D_\pi^m / F_\pi^0, \quad D_{\pi^{-1}}^{m-1} \simeq F_\pi^m / F_\pi^0.$$

Let us give the definition of an elementary G -module for the case of an arbitrary reductive Lie group G . Let N be a maximal unipotent subgroup of the group G , and B its normalizer in G . Consider an arbitrary finite-dimensional irreducible representation τ of the factor group B/N , acting in a space E , and lift it trivially to the group B . Let H_τ be the space of all infinitely differentiable functions $f(g)$ on G with values in E satisfying the following condition:

$$f(bg) = \tau(b)f(g)$$

for every $b \in B$. Define in H_τ a representation $T(g)$ of the group G by the formula

$$(T(g_0)f)(g) = f(gg_0).$$

The constructed G -modules H_τ will be called elementary. It is not difficult to show that, in the case of the group $SL(2, \mathbb{C})$, they coincide with the G -modules D_π^m defined above. **Hypothesis:** the G -modules H_τ are indecomposable.

3°. Classification of G -modules in the nonspecial case. We shall call a Harish-Chandra G -module nonspecial if all factors of its composition series are infinite-dimensional modules. It is obvious that all G -modules D_π^m , where π is a nonspecial point, are nonspecial. It can be proved

* The G -modules D_π^0 are the well-known construction of irreducible representations of Gel'fand–Naimark ⁽⁵⁾.

** If $n_1 > 0$, $n_2 > 0$, then the space F_π^m can be defined as follows. Fix a function $\varphi_m \in D_{\pi_0}^m$, $\pi_0 = (1, 1)$ ($\varphi_m \neq 0$), such that, for every $g \in G$,

$$T(g)\varphi_m = \varphi_m \pmod{D_{\pi_0}^{m-1}}$$

(for example, one may take $\varphi_m(z) = \ln^m(|z_1|^2 + |z_2|^2)$). We shall call a function $f \in D_\pi^m$ an m -th quasipolynomial if

$$f = P\varphi_m \pmod{D_\pi^{m-1}},$$

where P is a polynomial (it is not hard to see that the definition does not depend on the choice of φ_m). The space of m -th quasipolynomials is our F_π^m . We note that the space $F_{\pi^{-1}}^m$ can be defined from considerations of duality.

*** Both in the nonspecial and in the special case, D_π^m has only one composition series.

(3), see also (6), that these exhaust, up to equivalence, all indecomposable nonspecial Harish-Chandra G -modules. We note that two nonspecial G -modules $D_{\pi_1}^{m_1}$ and $D_{\pi_2}^{m_2}$ are equivalent if and only if $m_1 = m_2$ and either $\pi_2 = \pi_1$, or $\pi_2 = \pi_1^{-1}$.

4°. Elementary operations on G -modules. In the special case, indecomposable Harish-Chandra G -modules will be constructed from elementary ones by means of the following three elementary operations.

- a) **Gluing.** Let H_1, H_2 be two G -modules, $H'_1 \subset H_1$, $H'_2 \subset H_2$ isomorphic submodules, and let $a : H_1 \rightarrow H_2$ be a prescribed isomorphism. The gluing of H_1 and H_2 along the submodules H'_1, H'_2 will be the G -module $H(a) = (H_1 \oplus H_2)/H$, where H is the submodule consisting of pairs (x, ax) , $x \in H_1$. We shall call this operation **operation A**.
- b) **The dual operation.** Let H_1, H_2 be two G -modules; $H'_1 \subset H_1$, $H'_2 \subset H_2$ submodules such that there is an isomorphism between the quotient modules $a : H_1/H'_1 \rightarrow H_2/H'_2$. The gluing of H_1 and H_2 along the quotient modules H_1/H'_1 and H_2/H'_2 will be the submodule $\hat{H}(a) \subset H_1 \oplus H_2$ consisting of all pairs (h_1, h_2) , $h_1 \in H_1$, $h_2 \in H_2$, such that $a\bar{h}_1 = \bar{h}_2$, where $\bar{h}_i \in H_i/H'_i$ is the image of the element $h_i \in H_i$ under the natural homomorphism. We shall call this operation **operation B**.

Remark. Both operations are invariant under replacing a by λa , where $\lambda \neq 0$ is an arbitrary number. It follows from this that gluing along irreducible submodules (respectively along irreducible quotient modules) does not depend on the choice of the isomorphism a .

- c) **Polymerization.** Let H be a G -module and let $H_1 \neq H_2$ be two isomorphic submodules of it. Fix an isomorphism $a : H_1 \rightarrow H_2$. By the polymerization of m copies of the G -module H ($m = 1, 2, \dots$) we shall mean the G -module

$$H^{(m)}(\lambda, a) = \left(\bigoplus^m H \right) / H^\lambda,$$

where H^λ is the submodule consisting of the elements

$$(ah_1 - \lambda h_1, ah_2 - \lambda h_2 - h_1, \dots, ah_m - \lambda h_m - h_{m-1}), \quad (2)$$

where $h_1, \dots, h_m \in H_1$, $\lambda \neq 0$ is an arbitrary fixed complex number. It is easy to verify that $H^{(m)}(\lambda_0 \lambda, \lambda_0 a) = H^{(m)}(\lambda, a)$ for any $\lambda_0 \neq 0$.

5°. Classification of indecomposable G -modules in the special case. Consider a fixed special point $\pi = (n_1, n_2)$, where $n_1 > 0$, $n_2 > 0$. We list the special indecomposable Harish-Chandra G -modules having as composition-series factors F_π^0 and $F_{\pi-1}^0$.

- 1) Elementary modules: D_π^m , F_π^m , $D_{\pi-1}^m$, $F_{\pi-1}^{m+1*}$.
- 2) G -modules $D_\pi^{m_1, \dots, m_k}$, where $m_0 \geq 0$, $m_i > 0$ ($1 \leq i \leq k$), $k > 1$. They are constructed from k elementary modules:

$$F_{\pi-1}^{m_1}, D_{\pi'}^{m_2}, F_{\pi-1}^{m_3}, D_{\pi'}^{m_4}, \dots, \quad (3)$$

where modules of type $F_{\pi-1}^m$ stand in the odd positions, and modules of type $D_{\pi'}^m$ in the even positions ($\pi' = (n_1, -n_2)$). This construction is carried out as follows. In each of the modules of the chain (3) there are uniquely determined irreducible submodule and irreducible quotient module (both are isomorphic to $F_{\pi-1}^0$). The first two terms of the chain (3) are glued by operation B (see § 4, b) along irreducible quotient modules. We obtain the module $D_\pi^{m_1, m_2}$. In the natural way we associate to the irreducible submodule of $D_{\pi'}^{m_2}$ the irreducible submodule $H'_2 \subset D_\pi^{m_1, m_2**}$. We glue $D_\pi^{m_1, m_2}$ with $F_{\pi-1}^{m_3}$ by operation A (see § 4, a) along irreducible

* The module $F_{\pi-1}^0 \simeq D_{\pi'}^0$ is excluded, since by our classification this module is nonspecial. For the same reason, in the second group the modules $D_\pi^{0, m_2} \simeq D_{\pi'}^{m_2}$ are excluded.

** H'_2 consists of all pairs $(0, h) \in D_\pi^{m_1, m_2}$, where h ranges over the irreducible submodule of $D_{\pi'}^{m_2}$.

submodules $H'_2 \subset D_\pi^{m_1, m_2}$ and $F_{\pi-1}^0 \subset F_{\pi-1}^{m_3}$; we obtain the module $D_\pi^{m_1, m_2, m_3}$. In a natural way, to the irreducible quotient module $F_{\pi-1}^{m_3} / D_{\pi-1}^{m_3-1}$ we associate the irreducible quotient module $D_\pi^{m_1, m_2, m_3} / H'_3$. We glue $D_\pi^{m_1, m_2, m_3}$ with $D_{\pi'}^{m_4}$ by the operator B with respect to the irreducible quotient modules $D_\pi^{m_1, m_2, m_3} / H'_3$ and $D_{\pi'}^{m_4} / D_{\pi'}^{m_4-1}$; we obtain the module $D_\pi^{m_1, m_2, m_3, m_4}$, and so on.

- 3) The G -modules $D_{\pi,-+}^{m_1,\dots,m_k}$, $D_{\pi,+-}^{m_1,\dots,m_{2p+1}}$, $D_{\pi,--}^{m_1,\dots,m_{2p+1}}$, which in essence do not differ in any way from $D_{\pi}^{m_1,\dots,m_k}$. They are obtained by gluing the chain (3) in which one or both end terms have been shortened. Namely, in the construction of $D_{\pi,-+}^{m_1,\dots,m_k}$ the first term of the chain $F_{\pi-1}^{m_1}$ is replaced by the term $F_{\pi-1}^{m_1}/F_{\pi-1}^0 \simeq D_{\pi}^{m_1-1}$; in the construction of $D_{\pi,+-}^{m_1,\dots,m_{2p+1}}$ the last term $F_{\pi-1}^{m_{2p+1}}$ is replaced by the term $D_{\pi-1}^{m_{2p+1}-1} \subset F_{\pi-1}^{m_{2p+1}}$; finally, in the construction of $D_{\pi,--}^{m_1,\dots,m_{2p+1}}$ the shortenings are made at both ends of the chain (3).
- 4) The G -modules $D_{\pi}^{m_1,\dots,m_{2s};m,\lambda}$ (specified by a set of $2s$ ($s > 0$) integers $m_i > 0^{**}$, by one more integer $m > 0$, and by a complex number $\lambda \neq 0$). They are obtained by polymerizing m copies of the modules $D_{\pi}^{m_1,\dots,m_{2s}}$. Namely, in the G -modules $F_{\pi-1}^{m_1}$ and $D_{\pi'}^{m_{2s}}$, which are the end terms of the chain (3), irreducible submodules (isomorphic to $F_{\pi-1}^0$) are fixed. Let H_1, H_2 be the images of these submodules in $D_{\pi}^{m_1,\dots,m_{2s}}$ after gluing the chain (3). According to (3), the isomorphism $a: H_1 \rightarrow H_2$ can be defined canonically. We set

$$D_{\pi}^{m_1,\dots,m_{2s};m,\lambda} = (m \oplus D_{\pi}^{m_1,\dots,m_{2s}}) / H^{\lambda},$$

where H^{λ} is the submodule consisting of elements of the form (2).

On the sequences $(m_1, \dots, m_{2s})$ considered in this construction there is imposed the additional condition of aperiodicity: there is no divisor r of the number s ($r < s$) such that $m_p = m_q$ for any p and q congruent modulo $2r$.

Theorem. 1) The modules constructed above are indecomposable. 2) They are pairwise inequivalent, except for the case when both modules belong to the fourth class. Two modules from the fourth class $D_{\pi_1}^{m_1,\dots,m_{2s};m,\lambda}$ and $D_{\pi_2}^{m'_1,\dots,m'_{2s};m',\lambda'}$ are equivalent if and only if $\pi_1 = \pi_2$, $\lambda = \lambda'$, $m = m'$, $s = s'$, and $(m'_1, \dots, m'_{2s})$ is obtained from $(m_1, \dots, m_{2s})$ by a cyclic permutation by an even number of positions. 3) The constructed G -modules exhaust, up to equivalence, all special indecomposable Harish-Chandra G -modules.

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* H'_3 is the image of $D_{\pi}^{m_1, m_2} \oplus D_{\pi-1}^{m_3-1}$ ($F_{\pi-1}^{m_3}/D_{\pi-1}^{m_3-1}$ is an irreducible quotient module) under the natural mapping $D_{\pi}^{m_1, m_2} \oplus F_{\pi-1}^{m_3} \rightarrow D_{\pi}^{m_1, m_2, m_3}$.

** In the case $m_1 = 0$, gluing $F_{\pi-1}^0 \simeq D_{\pi}^0$ with $D_{\pi'}^{m_2}$ gives $D_{\pi'}^{m_2}$. Therefore, when $m_1 = 0$ one may assume that the chain (3) begins immediately with the term $D_{\pi'}^{m_2}$.

Note: Figure translations are in progress. See original paper for figures.

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