

ASYMPTOTICS OF THE SOLUTION OF A PROBLEM WITH AN INITIAL JUMP FOR SECOND-ORDER HYPERBOLIC EQUATIONS CONTAINING A SMALL PARAMETER

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Abstract

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MATHEMATICS

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ASYMPTOTICS OF THE SOLUTION OF A PROBLEM WITH AN INITIAL JUMP FOR SECOND-ORDER HYPERBOLIC EQUATIONS CONTAINING A SMALL PARAMETER

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1. Consider the Cauchy problem for hyperbolic equations of the form:

$$\varepsilon \left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} \right) = A(t, x, u) \frac{\partial u}{\partial t} + B(t, x, u), \quad (1)$$

$$u(t, x)|_{t=0} = \varphi(x), \quad \varepsilon \frac{\partial u}{\partial t} \Big|_{t=0} = \psi(x), \quad (2)$$

where $\varepsilon > 0$ is a small parameter. Suppose that in the strip $(t \geq 0, -\infty < x < +\infty)$, for all u ,

$$A(t, x, u) \geq \gamma > 0. \quad (3)$$

In the present paper it is proved that the solution u of problem (1), (2), as $\varepsilon \rightarrow 0$, will tend to the solution of the reduced equation

$$0 = A(t, x, u_0) \frac{\partial u_0}{\partial t} + B(t, x, u_0), \quad (4)$$

but, however, the solution of equation (4) does not satisfy the former initial condition (2), and satisfies an entirely different condition:

$$u_0(t, x)|_{t=0} = \varphi(x) + \Delta(x), \quad (5)$$

where the function $\Delta(x)$ will be called the initial jump of the function u , and it is determined uniquely from the following equation:

$$\psi(x) = \int_{\varphi(x)}^{\varphi(x)+\Delta(x)} A(0, x, u) du. \quad (6)$$

We now turn to the question of constructing the asymptotics, with respect to the small parameter ε , of the solution of problem (1), (2).

2. We shall seek an approximate solution of problem (1), (2) in the form of an expansion in integral powers of the small parameter ε

$$u(t, x, \varepsilon) = u_0(t, x) + \varepsilon u_1(t, x) + \dots + w_0(\tau, x) + \varepsilon w_1(\tau, x) + \dots, \quad (7)$$

where $\tau = t/\varepsilon$. Substituting expansion (7) into equation (1), we obtain:

$$\begin{aligned} & \varepsilon^2 \left(\frac{\partial^2 u_0(t, x)}{\partial x^2} + \varepsilon \frac{\partial^2 u_1(t, x)}{\partial x^2} + \dots + \frac{\partial^2 w_0(\tau, x)}{\partial x^2} + \varepsilon \frac{\partial^2 w_1(\tau, x)}{\partial x^2} + \dots \right) \\ & - \varepsilon^2 \left(\frac{\partial^2 u_0(t, x)}{\partial t^2} + \varepsilon \frac{\partial^2 u_1(t, x)}{\partial t^2} + \dots \right) - \left(\frac{\partial^2 w_0(\tau, x)}{\partial \tau^2} + \varepsilon \frac{\partial^2 w_1(\tau, x)}{\partial \tau^2} + \dots \right) \\ & = \left[A(\varepsilon\tau, x, u_0(\varepsilon\tau, x) + \varepsilon u_1(\varepsilon\tau, x) + \dots + w_0(\tau, x) + \varepsilon w_1(\tau, x) + \dots) \frac{\partial}{\partial \tau} (u_0(\varepsilon\tau, x) \right. \\ & \quad \left. + \varepsilon u_1(\varepsilon\tau, x) + \dots + w_0(\tau, x) + \varepsilon w_1(\tau, x) + \dots) \right. \\ & \quad \left. - A(\varepsilon\tau, x, u_0(\varepsilon\tau, x) + \varepsilon u_1(\varepsilon\tau, x) + \dots) \frac{\partial}{\partial \tau} (u_0(\varepsilon\tau, x) + \varepsilon u_1(\varepsilon\tau, x) + \dots) \right] \\ & \quad + \varepsilon A(t, x, u_0(t, x) + \\ & \quad + \varepsilon u_1(t, x) + \dots) \frac{\partial}{\partial t} (u_0(t, x) + \varepsilon u_1(t, x) + \dots) + \varepsilon [B(\varepsilon\tau, x, u_0(\varepsilon\tau, x) + \\ & \quad + \varepsilon u_1(\varepsilon\tau, x) + \dots + w_0(\tau, x) + \varepsilon w_1(\tau, x) + \dots) - B(\varepsilon\tau, x, u_0(\varepsilon\tau, x) + \\ & \quad + \varepsilon u_1(\varepsilon\tau, x) + \dots)] + \varepsilon B(t, x, u_0(t, x) + \varepsilon u_1(t, x) + \dots) \equiv \\ & \equiv [A(\varepsilon\tau, x, a_0(\tau, x) + w_0(\tau, x) + \varepsilon(a_1(\tau, x) + w_1(\tau, x)) + \dots) \frac{\partial}{\partial \tau} (a_0(\tau, x) + \\ & \quad + w_0(\tau, x) + \varepsilon(a_1(\tau, x) + w_1(\tau, x)) + \dots) - A(\varepsilon\tau, x, a_0(\tau, x) + \\ & \quad + \varepsilon a_1(\tau, x) + \dots) \frac{\partial}{\partial \tau} (a_0(\tau, x) + \varepsilon a_1(\tau, x) + \dots)] + \varepsilon A(t, x, u_0(t, x) + \\ & \quad + \varepsilon u_1(t, x) + \dots) \frac{\partial}{\partial t} (u_0(t, x) + \varepsilon u_1(t, x) + \dots) + \varepsilon [B(\varepsilon\tau, x, a_0(\tau, x) + \\ & \quad + w_0(\tau, x) + \varepsilon(a_1(\tau, x) + w_1(\tau, x)) + \dots) - B(\varepsilon\tau, x, a_0(\tau, x) + \\ & \quad + \varepsilon a_1(\tau, x) + \dots)] + \varepsilon B(t, x, u_0(t, x) + \varepsilon u_1(t, x) + \dots), \end{aligned} \quad (8)$$

where

$$a_k(\tau, x) = u_k(0, x) + \tau \frac{\partial u_{k-1}(0, x)}{\partial t} + \dots + \frac{\tau^k}{k!} \frac{\partial^k u_0(0, x)}{\partial t^k}, \quad a_k(0, x) = u_k(0, x).$$

From (8) we obtain two types of equations for determining the coefficients $u_k(t, x)$ and $w_k(\tau, x)$, $k \geq 0$, of the expansion (7). For $u_k(t, x)$ we have the following equation:

$$\begin{aligned} & \varepsilon \left(\frac{\partial^2 u_0}{\partial x^2} + \varepsilon \frac{\partial^2 u_1}{\partial x^2} + \dots \right) - \varepsilon \left(\frac{\partial^2 u_0}{\partial t^2} + \varepsilon \frac{\partial^2 u_1}{\partial t^2} + \dots \right) = \\ & = A(t, x, u_0 + \varepsilon u_1 + \dots) \left(\frac{\partial u_0}{\partial t} + \varepsilon \frac{\partial u_1}{\partial t} + \dots \right) + B(t, x, u_0 + \varepsilon u_1 + \dots), \end{aligned} \quad (9)$$

and for $w_k(\tau, x)$:

$$\begin{aligned} & \varepsilon^2 \left(\frac{\partial^2 w_0}{\partial x^2} + \varepsilon \frac{\partial^2 w_1}{\partial x^2} + \dots \right) - \left(\frac{\partial^2 w_0}{\partial \tau^2} + \varepsilon \frac{\partial^2 w_1}{\partial \tau^2} + \dots \right) = [A(\varepsilon \tau, x, a_0 + w_0 + \\ & + \varepsilon(a_1 + w_1) + \dots) \left(\frac{\partial}{\partial \tau}(a_0 + w_0) + \varepsilon \frac{\partial}{\partial \tau}(a_1 + w_1) + \dots \right) - \\ & - A(\varepsilon \tau, x, a_0 + \varepsilon a_1 + \dots) \left(\frac{\partial a_0}{\partial \tau} + \varepsilon \frac{\partial a_1}{\partial \tau} + \dots \right)] + \varepsilon [B(\varepsilon \tau, x, a_0 + w_0 + \\ & + \varepsilon(a_1 + w_1) + \dots) - B(\varepsilon \tau, x, a_0 + \varepsilon a_1 + \dots)]. \end{aligned} \quad (10)$$

For the unique determination of the coefficients u_k and w_k of the expansion (7), we prescribe the initial conditions in the following way:

$$u_0(t, x)|_{t=0} = \varphi(x) + \Delta(x), \quad (11)$$

$$w_0(\tau, x)|_{\tau=0} = \varphi(x) - u_0(0, x), \quad \frac{\partial w_0}{\partial \tau} \Big|_{\tau=0} = \psi(x), \quad (12')$$

$$w_k(\tau, x)|_{\tau=0} = -u_k(0, x), \quad \frac{\partial w_k}{\partial \tau} \Big|_{\tau=0} = -\frac{\partial u_{k-1}(0, x)}{\partial t}, \quad (12'')$$

and the initial conditions for $u_k(t, x)$ ($k > 0$) will be chosen in a special way (see below).

Expanding the right-hand sides of equations (9) and (10) in powers of ε and then equating coefficients of like powers of ε , we obtain a sequence of equations for $u_k(t, x)$ and $w_k(\tau, x)$:

$$0 = A(t, x, u_0) \frac{\partial u_0}{\partial t} + B(t, x, u_0), \quad (13')$$

$$\frac{\partial^2 w_0}{\partial \tau^2} + A(0, x, u_0(0, x) + w_0(\tau, x)) \frac{\partial w_0}{\partial \tau} = 0, \quad (13'')$$

$$A(t, x, u_0) \frac{\partial u_k}{\partial t} + \left[\frac{\partial A(t, x, u_0)}{\partial u} \frac{\partial u_0}{\partial t} + \frac{\partial B(t, x, u_0)}{\partial u} \right] u_k = \Phi_k(t, x), \quad (14')$$

$$\frac{\partial^2 w_k}{\partial \tau^2} + \frac{\partial}{\partial \tau} [A(0, x, \alpha_0 + w_0)(\alpha_k + w_k) - A(0, x, \alpha_0)\alpha_k] = \Psi_k(\tau, x), \quad (14'')$$

where the functions Φ_k and Ψ_k are expressed in terms of $u_i(t, x)$ and $\alpha_i(\tau, x)$, $w_i(\tau, x)$, $i < k$.

The initial conditions for $u_k(t, x)$ ($k > 0$) have not yet been determined. We shall choose them so that the function $w_k(\tau, x)$ and its first derivative with respect to τ are functions of boundary-layer type (1). In this connection, assuming that w_k , $\partial w_k / \partial \tau$, and $\Psi_k(\tau, x)$ are functions of boundary-layer type, while $A(t, x, u)$, $\partial A / \partial u$ are continuous and $|\partial A / \partial u| < c = \text{const}$ for $t \geq 0$, $-\infty < x < +\infty$, $-\infty < u < +\infty$, we integrate equation (14) with respect to τ from 0 to ∞ and, taking into account

$$\alpha_k(0, x) = u_k(0, x), \quad u_k(0, x) + w_k(0, x) = 0, \quad \left. \frac{\partial w_k}{\partial \tau} \right|_{\tau=0} = -\frac{\partial u_{k-1}(0, x)}{\partial t},$$

we obtain the initial condition for $u_k(t, x)$, $k > 0$:

$$u_k(t, x)|_{t=0} = \frac{1}{A(0, x, u_0(0, x))} \left(\int_0^\infty \Psi_k(\tau, x) d\tau - \frac{\partial u_{k-1}(0, x)}{\partial t} \right). \quad (15)$$

Suppose that in the strip $t \geq 0$, $-\infty < x < +\infty$, for all u the following conditions are satisfied:

a) derivatives of the form

$$\frac{\partial^p A(t, x, u)}{\partial t^{p_0} \partial x^{p_1} \partial u^{p_2}}, \quad \frac{\partial^p B(t, x, u)}{\partial t^{p_0} \partial x^{p_1} \partial u^{p_2}}, \quad (16)$$

are continuous, where $p = p_0 + p_1 + p_2$, $0 \leq p \leq N+1$, $0 \leq p_i \leq N+1$, $i = 0, 1, 2$;

b) the following are continuous:

$$\varphi^{(i)}(x), \quad \psi^{(j)}(x), \quad (17)$$

where $i = 0, 1, \dots, N + 1$, $j = 0, 1, \dots, N$.

The following assertions are valid.

Lemma 1. The solution $w_0(\tau, x)$ of problem (13), (12) and the derivatives of the form $\partial^{m+1} w_0 / \partial \tau \partial x^m$ are functions of boundary-layer type

$$|w_0(\tau, x)| \leq C_0(\tau, x)e^{-\gamma\tau}, \quad \left| \frac{\partial^{m+1} w_0}{\partial \tau \partial x^m} \right| \leq C_1(\tau, x)e^{-\gamma\tau}, \quad (18)$$

where $C_0(\tau, x)$ and $C_1(\tau, x)$ are polynomials in τ with bounded coefficients depending on x , $0 \leq m \leq N + 1$.

Proof. Denote $\partial w_0 / \partial \tau$ by $v_0(\tau, x)$. Then from (13''), (12') we have

$$v_0 = \psi(x) \exp \left(- \int_0^\tau A(0, x, u_0(0, x) + w_0) ds \right).$$

Hence, bearing (3) in mind, we obtain the estimate (18) for $v_0(\tau, x)$, while

$$w_0(\tau, x) = \varphi(x) - u_0(0, x) + \int_0^\tau v_0(s, x) ds,$$

and consequently, as $\tau \rightarrow \infty$, we obtain

$$w_0(\infty, x) = \varphi(x) - u_0(0, x) + \int_0^\infty v_0(s, x) ds,$$

where $\int_0^\infty v_0(s, x) ds$, by virtue of the estimate (18) for $v_0(\tau, x)$, converges. Integrating now equation (13'') and passing to the limit as $\tau \rightarrow \infty$, we obtain

$$\psi(x) = \int_{\varphi(x)}^{u_0(0, x) + w_0(\infty, x)} A(0, x, \xi) d\xi.$$

Hence, taking into account (6) and the expression for $w_0(\infty, x)$, we obtain

$$\Delta(x) = \int_0^\infty v_0(s, x) ds.$$

Then from the expression

$$w_0(\tau, x) = - \int_{\tau}^{\infty} v_0(s, x) ds$$

directly-

Consequently, estimate (18) follows for $w_0(\tau, x)$. By differentiating equation (13'') and using induction, one can verify the validity of (18) for $\partial^{m+1}w_0(\tau, x)/\partial\tau\partial x^m$, $1 \leq m \leq N+1$.

Lemma 2. *The solution $w_k(\tau, x)$ of problem (14''), (12'') and its derivatives of the form $\partial^{m+1}w_k/\partial\tau\partial x^m$, $1 \leq k \leq N+1$, $0 \leq m \leq N+1-k$, are functions of boundary-layer type.*

Lemma 2 is proved by the method of mathematical induction.

Theorem. *If conditions (3), (16), (17) are satisfied and, for $-\infty < u < +\infty$, $\partial A/\partial u$, $\partial B/\partial u$, $\partial^2 B/\partial u^2$ are bounded in Q (the characteristic triangle of equation (1)), then in Q there exists a unique^(2,3) solution of problem (1), (2), and it admits the following asymptotic expansion:*

$$u(t, x, \varepsilon) = \sum_{k=0}^N \varepsilon^k u_k(t, x) + \sum_{k=0}^{N+1} \varepsilon^k w_k\left(\frac{t}{\varepsilon}, x\right) + R_N(t, x, \varepsilon), \quad (19)$$

where $u_0(t, x)$ is the solution of equation (4) with initial condition (5), $u_k(t, x)$ is the solution of problem (14'), (15); $w_k(t/\varepsilon, x)$ are functions of boundary-layer type, constructed with the help of problems (13''), (12') and (14''), (12''). For R_N everywhere in Q the estimate holds

$$\|R_N\|_{L_2(Q)} = O(\varepsilon^{N+1}). \quad (20)$$

Remark. The investigation also carries over to the case $u(t, x_1, \dots, x_n)$.

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Note: Figure translations are in progress. See original paper for figures.

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