

ASYMPTOTIC PROPERTIES OF THE GREEN' S FUNCTIONS OF A NONIDEAL BOSE GAS

Yu. A. Tserkovnikov

1970

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-197001.24129>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 537.312.62

PHYSICS

Yu. A. TSERKOVNIKOV

ASYMPTOTIC PROPERTIES OF THE GREEN'S FUNCTIONS OF A NONIDEAL BOSE GAS

(Presented by Academician N. N. Bogolyubov on 23 VI 1969)

In work ⁽¹⁾ it was shown that in the theory of a nonideal Bose gas it is convenient to use a method of decoupling chains of equations for Green's functions based on projecting second-quantized operators in a space with scalar product

$$(A, B) = -\langle\langle A | B^+ \rangle\rangle_{E=0}, \quad (1)$$

where

$$\langle\langle A | B^+ \rangle\rangle_E = \int_{-\infty}^{\infty} e^{iEt} \langle\langle A(t); B^+ \rangle\rangle dt \quad (2)$$

is the Fourier component of the Green's function

$\langle\langle A(t); B^+ \rangle\rangle = -i\theta(t)\langle[A(t), B^+]\rangle$, and

$A(t) = \exp\{iHt\}A\exp\{-iHt\}$ is the operator in the Heisenberg representation for a system with Hamiltonian H ⁽²⁾. In the present work it will be shown that this method is also useful in the investigation of the asymptotics of the Green's functions of a nonideal Bose gas for small values of the energy and momentum of a particle. In this case, for the one-particle Green's functions we obtain poles corresponding to ordinary and second sound, without resorting to the usual construction of a kinetic equation in these cases (see, for example, ⁽³⁾).

We shall consider the case of low temperatures, when the number of particles in the condensate N_0 is, in order of magnitude, equal to the total number of particles N , and the interaction between particles is small. In accordance with this, in the equations of motion for the annihilation and creation operators $a_q(t)$ and $a_{-q}^+(t)$ we introduce a formal small parameter ε , replacing the Fourier component of the interaction potential $v(k)$ by $\varepsilon v(k)$, and N and N_0 by $\varepsilon^{-1}N$ and $\varepsilon^{-1}N_0$. The equation of motion for $a_q(t)$ has the form

$$i \frac{da_q}{dt} = [a_{qH}] = \left(\frac{q^2}{2m} - \mu + \frac{N}{V} v(0) \right) + \frac{N_0}{V} v(q) (a_q + a_{-q}^+) + \sqrt{\varepsilon} A_q + \frac{\varepsilon}{V} \sum_{k \neq 0, q; p \neq 0, -k} v(k) a_p^+ a_{p+k} a_{-k+q}, \quad (3)$$

where

$$A_q = \sqrt{\varepsilon} \frac{\sqrt{N_0}}{V} \sum \{ (v(q) + v(k)) a_{-k}^+ + v(k) a_k \} a_{-k+q};$$

a_q , a_p , a_k , etc. are Bose creation and annihilation operators for particles in a state with momentum $\neq 0$; V is the volume of the system. The chemical potential μ is determined from the condition $\langle a_q \rangle = \delta_{q0} \sqrt{N_0}$ and is equal to

$$\mu = \frac{N}{V} v(0) + \frac{\varepsilon}{V} \sum v(k) \langle (a_k^+ + a_{-k}) a_k \rangle + \frac{\varepsilon^{3/2}}{\sqrt{N_0} V} \sum v(k) \langle a_p^+ a_{p+k} a_{-k} \rangle. \quad (4)$$

For $\varepsilon = 0$ we obtain the result of work ⁽⁴⁾, which is expressed by the relations

$$a_q(t) = u_q \alpha_q e^{iE_q t} + v_q \alpha_{-q}^+ e^{iE_q t}, \quad E_q = \sqrt{\left(\frac{q^2}{2m} \right)^2 + \frac{q^2}{m} \frac{N_0}{V} v(q)}. \quad (5)$$

where a_q and a_{-q}^+ are Bose operators of annihilation and creation of quasiparticles with momentum q and energy E_q , and in the zeroth approximation

$$n_q = \langle a_q^+ a_q \rangle = (e^{E_q/\theta} - 1)^{-1}, \quad \langle a_{-q} a_q \rangle = 0. \quad (6)$$

The parameters u_q and v_q are related by

$$u_q^2 - v_q^2 = 1, \quad u_q^2 + v_q^2 = \left(\frac{q^2}{2m} + \frac{N_0}{V} v(q) \right) E_q^{-1}, \quad u_q v_q = -\frac{N_0}{2V} \frac{v(q)}{E_q}. \quad (7)$$

Taking into account terms of higher order, in addition to the assumption made about the smallness of the interaction characterized by the parameter ε , we shall assume that the range of the forces is large—greater than the mean distance between particles, equal to $(V/N)^{1/3}$. In this case the Fourier components of the potential $\nu(k)$ under the summation sign will cut out a small region near $k = 0$, and as $V \rightarrow \infty$ these terms may be neglected. As a result, for the one-particle Green functions (their Fourier components (2)) we shall have the equations

$$E\langle\langle\varphi_q | a_q^+\rangle\rangle = 1 + \langle\langle\pi_q | a_q^+\rangle\rangle,$$

$$E\langle\langle\pi_q | a_q^+\rangle\rangle = \frac{q^2}{2m} + E_q^2\langle\langle\varphi_q | a_q^+\rangle\rangle + 2\sqrt{\varepsilon}\sqrt{N_0}\frac{\nu(q)}{V} \sum_{k \neq 0, q} \langle\langle a_{-k}^+ a_{-k+q} | a_q^+\rangle\rangle, \quad (8)$$

where the operators φ_q and π_q are equal to $\varphi_q = a_q + a_{-q}^+$, $\pi_q = (q^2/2m)(a_q - a_{-q}^+)$, and possess the property $\varphi_q^+ = \varphi_{-q}$, $\pi_q^+ = -\pi_{-q}$. Inclusion in equations (8) of the discarded terms would not change their qualitative character and would lead only to a renormalization of the mass m and the potential $\nu(k)$ (see, for example, (1)).

The Green function $\langle\langle \sum a_{-k}^+ a_{-k+q} | a_q^+ \rangle\rangle$, entering the right-hand side of the last of equations (8), is a collective variable. In a nonideal Bose gas the conserved quantities are the flux and the total energy of the quasiparticles. Therefore, in order to take into account the collective properties of the system, we supplement the basis of operators φ_q and π_q , used in (1), with the operators

$$g_q = \sum_{k \neq 0, \pm q/2} k \cdot q a_{k-q/2}^+ a_{k+q/2}, \quad h_q = \frac{1}{2} \sum_{k \neq 0, \pm q/2} (E_{k-q/2} + E_{k+q/2}) a_{k-q/2}^+ a_{k+q/2}, \quad (9)$$

which are the divergence of the flux density and the energy density of the quasiparticles. Here $g_q^+ = -g_{-q}$ and $h_q^+ = h_{-q}$. We note that the total number of quasiparticles is not conserved. Projecting the operator $\sum a_{k-q/2}^+ a_{k+q/2}$ onto the chosen subspace, we shall have

$$\sum_{k \neq 0, \pm q/2} a_{k-q/2}^+ a_{k+q/2} \simeq \left(\sum a_{k-q/2}^+ a_{k+q/2}, h_q \right) (h_q, h_q)^{-1} \cdot h_q. \quad (10)$$

The remaining projections either are equal to zero or give a small contribution to the renormalization of the mass and the potential. The scalar products on the right-hand side of (10) are computed in the approximation of free quasiparticles, defined by relations (5), (6), and (7). Using definition (1), we find

$$\begin{aligned} \left(\sum a_{k-q/2}^+ a_{k+q/2}, h_q \right) &\simeq \sum (u_k^2 + v_k^2) \frac{\langle [a_{k-q/2}^+ a_{k+q/2}, h_q^+] \rangle}{E_{k-q/2} - E_{k+q/2}} \simeq \\ &\simeq - \sum_{k \neq 0} (u_k^2 + v_k^2) E_k \frac{\partial n_k}{\partial E_k} = \frac{1}{\theta} \sum_{k \neq 0} \left\{ \frac{k^2}{2m} + \frac{N_0}{V} \nu(k) \right\} n_k (1 + n_k) \simeq \end{aligned}$$

$$\simeq \frac{1}{\theta} \sum_{k \neq 0} \frac{k^2}{2m} n_k (1 + n_k), \quad (11)$$

where we have assumed that the momentum q is small and in the last equality neglected the term containing $\nu(k)$ under the summation sign. Similarly we obtain

$$(h_q, h_q) \simeq \sum E_k^2 \langle a_{k-q/2}^+ a_{k+q/2}, a_{k'+q/2}^+ a_{k'-q/2} \rangle \simeq \frac{1}{\theta} \sum E_k^2 n_k (1 + n_k). \quad (12)$$

It remains for us now to construct equations for the collective Green's functions $\langle\langle g_q | a_q^+ \rangle\rangle$ and $\langle\langle h_q | a_q^+ \rangle\rangle$. To this end, using equation (3) and the expression of α_q in terms of the operators a_q and a_{-q}^+ : $\alpha_q = u_q a_q - v_q a_{-q}^+$, we write the equation of motion for the operator $\alpha_{k-q/2}^+ \alpha_{k+q/2}$

$$i \frac{d}{dt} \alpha_{k-q/2}^+ \alpha_{k+q/2} = (E_{k+q/2} - E_{k-q/2}) \alpha_{k-q/2}^+ \alpha_{k+q/2} + \sqrt{\varepsilon} (\alpha_{k-q/2}^+ A_{k+q/2} - A_{k-q/2}^+ \alpha_{k+q/2}) \quad (13)$$

On the right-hand side of equation (13) we perform all possible contractions of operators. This is equivalent to projecting the last term in (13) onto the subspace spanned by the operators φ_q and π_q (allowance for scalar products with the operators g_q and h_q gives a contribution $\sim \sqrt{\varepsilon}$). In addition, we retain only those terms that contain $v(q)$, so that in the subsequent summation over k the potential is outside the summation sign. As a result, for the collective Green's functions we shall have the equations

$$\begin{aligned} E \langle\langle g_q | a_q^+ \rangle\rangle &= i \sum (k \cdot q) (E_{k+q/2} - E_{k-q/2}) \langle\langle \alpha_{k-q/2}^+ \alpha_{k+q/2} | a_q^+ \rangle\rangle + \\ &+ \sqrt{\varepsilon} \frac{\sqrt{N_0}}{V} v(q) \sum (u_{k-q/2} u_{k+q/2} + v_{k-q/2} v_{k+q/2}) (k \cdot q) (n_{k-q/2} - n_{k+q/2}) \langle\langle \varphi_q | a_q^+ \rangle\rangle, \\ E \langle\langle h_q | a_q^+ \rangle\rangle &= \frac{1}{2} \sum (E_{k+q/2} + E_{k-q/2}) (E_{k+q/2} - E_{k-q/2}) \langle\langle \alpha_{k-q/2}^+ \alpha_{k+q/2} | a_q^+ \rangle\rangle. \end{aligned} \quad (14)$$

The last term of the first equation for small q can be represented in the form

$$\frac{\sqrt{N_0}}{V} v(0) \sum_{k \neq 0} (u_k^2 + v_k^2) (k \cdot q) \frac{\partial E_k}{\partial k} \frac{(k \cdot q)}{k} \frac{\partial n_k}{\partial E_k} = -\frac{q^2}{3\theta} \frac{\sqrt{N_0}}{V} v(0) \times$$

$$\times \sum \left(\frac{k^2}{2m} + \frac{N_0}{V} v(k) \right) E_k^{-1} \frac{\partial E_k}{\partial k} k n_k (1+n_k) \simeq -\frac{q^2}{3\theta} \frac{\sqrt{N_0}}{V} v(0) \sum \frac{k^2}{2m} n_k (1+n_k), \quad (15)$$

where we have also used the fact that at low temperatures one may put

$$\frac{\partial E_k}{\partial k} k \simeq \sqrt{\frac{N_0}{V} \frac{v(0)}{m}} k \simeq E_k$$

and have neglected the term containing $v(k)$.

Let us express the two-particle operators in the right-hand sides of equations (14) in terms of g_q and h_q :

$$\begin{aligned} & \sum_{k \neq 0, \pm q/2} (k \cdot q) (E_{k+q/2} - E_{k-q/2}) \alpha_{k-q/2}^+ \alpha_{k+q/2} \simeq \\ & \simeq \sum (k \cdot q) (E_{k+q/2} - E_{k-q/2}) (\alpha_{k-q/2}^+ \alpha_{k+q/2}, h_q) (h_q, h_q)^{-1} h_q; \end{aligned} \quad (16)$$

$$\begin{aligned} & \frac{1}{2} \sum_{k \neq 0, \pm q/2} (E_{k+q/2} + E_{k-q/2}) (E_{k+q/2} - E_{k-q/2}) \alpha_{k-q/2}^+ \alpha_{k+q/2} \simeq \\ & \simeq \sum (E_{k+q/2} + E_{k-q/2}) (E_{k+q/2} - E_{k-q/2}) (\alpha_{k-q/2}^+ \alpha_{k+q/2}, g_q) (g_q, g_q)^{-1} g_q. \end{aligned} \quad (17)$$

The scalar products entering the right-hand sides of (16) and (17) are equal to

$$\begin{aligned} & \sum_{k \neq 0, \pm q/2} (k \cdot q) (E_{k+q/2} - E_{k-q/2}) (\alpha_{k-q/2}^+ \alpha_{k+q/2}, h_q) \simeq \\ & \simeq \frac{1}{2} \sum_{k \neq 0, \pm q/2} (E_{k+q/2} + E_{k-q/2}) (E_{k+q/2} - E_{k-q/2}) (\alpha_{k-q/2}^+ \alpha_{k+q/2}, g_q) \simeq \\ & \simeq -\frac{q^2}{3} \sum_{k, k' \neq 0} E_k \frac{\partial E_k}{\partial k} k (\alpha_{k+q/2}^+ \alpha_{k-q/2}, \alpha_{k'+q/2}^+ \alpha_{k'-q/2}), \end{aligned} \quad (18)$$

$$(g_q, g_q) \simeq \frac{q^2}{3} \sum_{k, k' \neq 0} k^2 (\alpha_{k+q/2}^+ \alpha_{k-q/2}, \alpha_{k'+q/2}^+ \alpha_{k'-q/2}). \quad (19)$$

The scalar product (h_q, h_q) was calculated in (12).

At low temperatures ($\theta \rightarrow 0$), the main contribution to the sum is made by terms with small k . Therefore, in (18) and (19) $\frac{\partial E_k}{\partial k} k \simeq E_k \simeq \sqrt{\frac{N_0 v(0)}{V} \frac{1}{m}} k$. As a result, equations (14) are written in the form

$$E \langle\langle g_q | a_q^+ \rangle\rangle = -\frac{q^2}{3} \langle\langle h_q | a_q^+ \rangle\rangle - \sqrt{\varepsilon} \sqrt{N_0} \frac{q^2}{3\theta} \frac{v(0)}{V} \sum_{k \neq 0} \frac{k^2}{2m} n_k (1 + n_k) \langle\langle \varphi_q | a_q^+ \rangle\rangle, \quad (20)$$

$$E \langle\langle h_q | a_q^+ \rangle\rangle = -\frac{N_0}{V} \frac{v(0)}{m} \langle\langle g_q | a_q^+ \rangle\rangle,$$

whence we find

$$\langle\langle h_q | a_q^+ \rangle\rangle = \sqrt{\varepsilon} \sqrt{N_0} \frac{q^2}{3} C_0^2 \left\{ E^2 - \frac{q^2}{3} C_0^2 \right\}^{-1} \lambda \langle\langle \varphi_q | a_q^+ \rangle\rangle, \quad (21)$$

where

$$\lambda = \frac{v(0)}{\theta v} \left(\sum_{k \neq 0} \frac{k^2}{2m} n_k (1 + n_k) \right)^2 \left(\sum_{k \neq 0} E_k^2 n_k (1 + n_k) \right)^{-1} \quad (22)$$

and $C_0^2 = \frac{N_0}{V} \frac{v(0)}{m}$. It is easy to see that $\lambda \rightarrow 0$ as $\theta \rightarrow 0$.

Substituting (10) and (22) into equations (8), we find the one-particle Green' s functions. For small E and q we shall have

$$\langle\langle a_q | a_q^+ \rangle\rangle \simeq -\langle\langle a_{-q}^+ | a_q^+ \rangle\rangle \simeq \frac{N_0}{V} v(0) \frac{E^2 - \frac{1}{3} C_0^2 q^2 (1 - \lambda)}{E^4 - \frac{4}{3} E^2 C_0^2 q^2 + \frac{1}{3} C_0^4 q^4 (1 - \lambda)}. \quad (23)$$

The Green' s functions (23) have poles corresponding to first sound and second sound,

$$E_q^{(1)} = \pm C_0 q \left(1 + \frac{1}{4} \lambda \right), \quad E_q^{(2)} = \pm \frac{1}{\sqrt{3}} C_0 q \left(1 - \frac{3}{4} \lambda \right). \quad (24)$$

Expressions (23), obtained within the framework of the theory of a weakly non-ideal Bose gas, are the analogue of the Green' s functions found in work (5) on the basis of the hydrodynamics of a superfluid liquid.

In conclusion, I express my deep gratitude to Acad. N. N. Bogolyubov for his attention to the work.

Received
20 V 1969

CITED LITERATURE

1. Yu. A. Tserkovnikov, DAN, **190**, No. 4 (1970).
2. D. N. Zubarev, UFN, **71**, 71 (1960).
3. Li Chzhen-Chzhun, DAN, **169**, 1054 (1966).
4. N. N. Bogolyubov, Izv. AN SSSR, ser. fiz., **11**, 67 (1947).
5. N. N. Bogolyubov, *On the Question of the Hydrodynamics of a Superfluid Liquid*, Preprint, Joint Institute for Nuclear Research, 1963.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.