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Abstract

Full Text

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MATHEMATICS

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AN n -DIMENSIONAL CUBE CANNOT BE DECOMPOSED INTO A COUNTABLE UNION OF PROPER CLOSED SUBSETS WHOSE PAIRWISE INTERSECTIONS ARE AT MOST $(n - 2)$ -DIMENSIONAL

(Presented by Academician P. S. Aleksandrov, 14 IV 1970)

The assertion announced in the title, for two summands, was proved by P. S. Urysohn in his famous paper "On Cantorian manifolds" (see ⁽¹⁾, p. 435).

Let an n -dimensional parallelepiped P be contained in a countable union of closed subsets Φ_k of the space E^n , $P \not\subset \Phi_k$, and $\dim(\Phi_i \cap \Phi_j) \leq n - 2$ for any distinct i and j .

Lemma 1. *For every i , at least one of the summands Φ_k contains a continuum joining the opposite faces P_{-i} and P_{+i} of the parallelepiped P .*

By symmetry, it suffices to restrict ourselves to the case $i = 1$. Put

$$M = \bigcup_{i \neq j} (\Phi_i \cap \Phi_j).$$

The set M has dimension $\dim M \leq n - 2$, while (P_{-i}, P_{+i}) , $i = 2, 3, \dots, n$, are $(n - 1)$ pairs of closed sets for which $P_{-i} \cap P_{+i} = \emptyset$. Then in P there exist closed partitions C_i between P_{-i} and P_{+i} such that the set

$$C = \bigcap_{i=2}^n C_i$$

does not intersect the set M .

To prove the lemma, it is enough to find at least one component of connectedness K of the compactum C joining P_{-1} with P_{+1} . Indeed, since $K \cap M = \emptyset$, the continuum K is decomposed into a union of pairwise disjoint closed sets $K \cap \Phi_k$. Hence, by the well-known theorem of Sierpiński ⁽²⁾, $K = K \cap \Phi_k$ for some k , i.e. $K \subset \Phi_k$, as required by the lemma. Let us show that such a component K can be found. Indeed, if there were no such components, then,

by a known assertion that every component of connectedness of a compactum is its quasicomponent, we could, for every component of connectedness of the compactum C , take a clopen neighborhood O in C not intersecting both faces P_{-1} and P_{+1} at the same time. These neighborhoods O cover the compactum C , and from them one can choose a finite subcover $O_1, O_2, \dots, O_p$. Denote by C_- the union of the face P_{-1} with all those neighborhoods O_i which do not intersect the face P_{+1} , and by C_+ the union of the face P_{+1} with the complement $C \setminus C_-$, which is clopen in C . It is clear that the closed sets C_- and C_+ do not intersect. Hence they can be separated in the parallelepiped P by a closed partition C_1 . Thus the opposite faces of the parallelepiped P would be separated by partitions $C_1, C_2, \dots, C_n$ with empty intersection, which is impossible. Consequently, contrary to the supposition, there exists a component of connectedness K of the compactum C joining the face P_{-1} with the face P_{+1} , which was to be proved.

Lemma 2. *No summand Φ_k contains any parallelepiped N two faces of which are parallel to the faces P_{-i} and P_{+i} , while the remaining ones lie on the faces of the parallelepiped P .*

Of course, we may again assume that $i = 1$. Suppose that, for example, the summand Φ_1 contains entirely some parallelepiped N of the indicated...

of the prescribed form. Then the union of the closed differences $\Phi_k \setminus \text{Int } N$, $k \neq 1$, together with the first summand Φ_1 , covers the parallelepiped P ; none of them contains P , and all their pairwise intersections are at most $(n - 2)$ -dimensional. Therefore one may assume that the given decomposition is such that a certain parallelepiped N of the indicated form does not meet any of the summands Φ_k , $k \neq 1$. Since the closed set Φ_1 does not contain P , there is in P a cube Q , with faces parallel to the faces of the parallelepiped P , which does not meet the set Φ_1 . Extending its faces, parallel to the faces P_i , $i = 2, 3, \dots, n$, up to their intersection with the faces P_{-1} and P_{+1} , we obtain a parallelepiped R which contains Q . The parallelepiped R does not lie in any of the summands Φ_k , $k \neq 1$, since otherwise N would meet one of them. Nor does R lie in Φ_1 , since the cube Q does not lie in Φ_1 . Hence, by Lemma 1, there exists a continuum K lying in some $R \cap \Phi_k$ and joining the faces of the parallelepiped R that lie on the faces P_{-1} and P_{+1} . This continuum necessarily intersects the parallelepiped N , and consequently $\Phi_k \cap N \neq \emptyset$, which means that $k = 1$. Thus $K \subset \Phi_1$. But the continuum K also intersects the cube Q , so that $Q \cap \Phi_1 \neq \emptyset$, and this contradicts the choice of the cube Q . The lemma is proved.

Theorem. *An n -dimensional cube cannot be decomposed into a countable union of closed subsets distinct from the whole cube, whose pairwise intersections are no more than $(n - 2)$ -dimensional.*

Let the n -dimensional cube I^n be decomposed into a countable union of closed subsets Φ_k , $\Phi_k \neq I^n$, with $\dim(\Phi_i \cap \Phi_j) \leq n - 2$ for any distinct i and j . Since I^n is a complete metric space, all summands Φ_k cannot be nowhere dense. Hence at least one of the summands contains entirely some cube Q with faces parallel to the faces of the cube I^n . Let this be the summand Φ_1 . There exists a

sequence of parallelepipeds P_i , $i = 1, 2, \dots, n+1$, with faces parallel to the faces of the cube I^n , the first of which coincides with I^n , and the last with Q , and, moreover, all faces of the parallelepiped P_{i+1} , except for one pair of opposite ones, lie on faces of the parallelepiped P_i . We obtain the parallelepiped P_n as follows. Extend all faces of the cube Q , except for one single pair of opposite faces, until they meet the faces of the cube I^n . The extended faces divide the cube I^n into 3^{n-1} parallelepipeds. P_n will be the one among them that contains Q . We construct the parallelepiped P_{n-1} in the same way, but this time starting from P_n and extending all its faces except for one single pair of opposite faces, different from those that lie on the faces of the cube I^n . It is clear that this process will stop at the $(n-1)$ -st step, at which we obtain the parallelepiped P_2 . All its faces, except for one single pair of opposite ones, will lie on faces of the cube I^n .

By Lemma 2, the parallelepiped P_2 is not contained entirely in any of the summands Φ_k . Then, by Lemma 2, P_3 is not contained in any of the summands Φ_k . Continuing this reasoning, at the n -th step we obtain that Q is not contained in any of the summands Φ_k , which contradicts the choice of the cube Q . The theorem is proved.

Corollary 1. *If a separable topological vector space L has a basis consisting of at least n elements, then L cannot be decomposed into a countable union of closed subsets, distinct from the whole L , whose pairwise intersections are no more than $(n-2)$ -dimensional.*

This corollary is obtained quite easily from the theorem with the aid of the following lemma:

Lemma 3. *Let $\{X_\alpha\}$ be a family of such closed subsets of a space X that, for every α , X_α cannot be decomposed into a countable union of closed subsets distinct from X_α , whose pairwise intersections are no more than $(n-1)$ -dimensional; then, if $\dim(X_{\alpha_0} \cap X_\alpha) \geq n$ for every $\alpha \neq \alpha_0$, then $X' = \bigcup_\alpha X_\alpha$ also cannot be decomposed into a countable union of the indicated form.*

Suppose there exists a decomposition $X' = \bigcup_{k=1}^\infty \Phi_k$, where the summands Φ_k are closed, $\Phi_k \neq X'$ for every k , and $\dim(\Phi_i \cap \Phi_j) \leq n-1$ for distinct i and j . Since X_α is not decomposed in this way, for every α there exists a Φ_{k_α} containing X_α . From the inclusion $X_{\alpha_0} \cap X_\alpha \subset \Phi_{k_{\alpha_0}} \cap \Phi_{k_\alpha}$ it follows that $\dim(\Phi_{k_{\alpha_0}} \cap \Phi_{k_\alpha}) \geq n$, and this entails $k_\alpha = k_{\alpha_0}$, i.e. $\Phi_{k_\alpha} = \Phi_{k_{\alpha_0}}$. Then $X_\alpha \subset \Phi_{k_{\alpha_0}}$ for every α and, consequently, $X' = \Phi_{k_{\alpha_0}}$, which is a contradiction. The lemma is proved.

Proof of Corollary 1. Let $l_1, l_2, \dots, l_n$ be distinct basis elements of the space L , and let a be an arbitrary element of the space L . Denote by L_a the convex hull spanned by the elements $a, l_1, l_2, \dots, l_n$. L_a is a closed subset of the space L , homeomorphic to an m -dimensional simplex T^m , where $m \geq n$. By the theorem, the set L_a is not decomposed into a countable union of closed subsets different from L_a , the pairwise intersections of which are at most $(n-2)$ -dimensional.

Moreover, the intersection $L_{l_1} \cap L_a$ contains L_{l_1} , so that $\lim(L_{l_1} \cap L_a) \geq n - 1$. Finally, $L = \bigcup_{\alpha} L_{\alpha}$. To complete the proof of Corollary 1, it remains only to apply Lemma 3.

Corollary 2. *If M is a connected topological space, every point x of which has a neighborhood homeomorphic to the $m(x)$ -dimensional Euclidean space $E^{m(x)}$, where $m(x) \geq n$, then M is not decomposed into a countable union of closed subsets different from all of M , the pairwise intersections of which are at most $(n - 2)$ -dimensional.*

Let $M = \bigcup_{k=1}^{\infty} \Phi_k$, where $M \neq \Phi_k$ for every k , $\dim(\Phi_i \cap \Phi_j) \leq n - 2$ whenever $i \neq j$, and the sets Φ_k are closed. Let $x \in M$ and let $\overline{O_x}$ be a closed neighborhood of the point x , homeomorphic to the cube $I^{m(x)}$, where $m(x) \geq n$. From the theorem it follows that $\overline{O_x}$ necessarily lies in one of the summands Φ_k , say in Φ_1 . Hence $U = \text{Int } \Phi_1 \neq \emptyset$. Since the space M is connected, $\text{Fr } U \neq \emptyset$. Now let $y \in \text{Fr } U$, and let $\overline{O_y}$ be a closed neighborhood of the point y , homeomorphic to the cube $I^{m(y)}$, where $m(y) \geq n$. There will be a summand Φ_k , $k \neq 1$, such that $\overline{O_y} \subset \Phi_k$. Put $F_1 = \Phi_1 \cap \overline{O_y}$ and $F_2 = \overline{O_y} \setminus U$. It is clear that $F_1 \neq \overline{O_y}$, $F_2 \neq \overline{O_y}$, $F_1 \cup F_2 = \overline{O_y}$, and $\dim(F_1 \cap F_2) \leq \dim(\Phi_1 \cap \Phi_k) \leq n - 1$, which contradicts the theorem. Consequently, a decomposition of the space M of the indicated kind is impossible.

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Note: Figure translations are in progress. See original paper for figures.

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